Enlarging Properties of Graphs

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Abstract

The subjects of this Thesis are the enlarging and magnifying properties of graphs. Upper bounds for the isoperimetric number $i(G)$ of a graph G are determined with respect to such elementary graph properties as order, valency, and the number of three and four-cycles. The relationship between $i(G)$ and the genus of G is studied in detail, and a class of graphs called finite element graphs is shown never to supply enlarging families. The magnifying properties of Hamiltonian cubic graphs are investigated, and a class of graphs known as shift graphs is defined. These are shown never to form enlarging families, using a technical lemma derived from Klawe's Theorem on non-expanding families of graphs. The same lemma is used, in conjunction with some elementary character theory, to prove that several important classes of Cayley graphs do not form enlarging families, and to derive a lower bound on the subdominant eigenvalue of a vertex-transitive graph. The problem of finding Ramanujan graphs is discussed. Some necessary conditions for a graph to be Ramanujan, depending on the automorphism group of the graph, and the number of certain reduced walks in the graph, are derived. Finally, the techniques of Buck are used to construct an infinite number of families of linear expanders, deploying free subgroups of the group $SL(2, \mathbf{Z})$.

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Chapter One

Introduction

1.1 Basic Definitions

In what follows we will assume, unless otherwise stated, that all graphs have no loops or multiple edges (such graphs will be called *simple*), and are connected and undirected. The notation $G = (V, E)$ denotes a graph G with vertex set V and edge set E. The *order* of G is the number of vertices.

For a subset $X \subseteq V$ we define the *isoperimetric number* of X to be

$$i(X) = \frac{|\Gamma(X) \setminus X|}{|X|},$$

where

$$\Gamma(X) = \{\, v \in V \mid \exists x \in X \quad \text{with} \quad \{x, v\} \in E \,\},$$

and $\Gamma(X) \setminus X$ denotes the set of elements of $\Gamma(X)$ that are not also in X. From this we have the notion of a *magnifier*.

Definition 1.1. *An (n, k, c)-magnifier is a k-regular graph $G = (V, E)$ of order n with the property that $i(X) \geq c$ for all $X \subseteq V$ such that $|X| \leq n/2$.* ▪

We define the *isoperimetric number* $i = i(G)$ of G to be

$$i = \min\{\, i(X) \mid X \subseteq V, |X| \leq n/2 \,\}.$$

The *adjacency matrix* of an undirected (but not necessarily simple) graph $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$ is the $n \times n$ matrix $A = [a_{ij}]$, where a_{ij} is defined to be the number of edges between v_i and v_j in G. The *spectrum* of G is simply that of A. (For results linking the spectral and structural properties of a graph, see [**Bi**]). If G is a k-regular graph which is undirected, then all its eigenvalues will be real and the largest will equal k. The condition that the multiplicity of k is equal to one is equivalent to the connectedness of G. Once the single eigenvalue k has been removed from the spectrum of G then the largest remaining eigenvalue is called the *subdominant eigenvalue* of G, and is denoted by $\lambda_1(G)$. Hence, $\lambda_1(G) \neq k$ is equivalent to the connectedness of G.

Definition 1.2. *An (n, k, ϵ)-enlarger ($\epsilon > 0$) is a k-regular graph G of order n such that $\lambda_1 \leq k - \epsilon$.* ∎

The close link between magnifiers and enlargers is revealed in the following lemma, due to Alon.

Lemma 1.3. *(i) Every (n, k, ϵ)-enlarger is an (n, k, c)-magnifier with $c = 2\epsilon/(k + 2\epsilon)$.*

(ii) Every (n, k, c)-magnifier is an (n, k, ϵ)-enlarger with $\epsilon = c^2/(4 + 2c^2)$.

Proof. (i) [**Al**, Corollary 2.3].

(ii) [**Al**, Lemma 2.4]. ∎

One might expect that, as the order of the graph becomes larger (with k fixed), the eigenvalue $\lambda_1(G)$ would approach the valency k, but there exist families where this does not happen. A *family of linear enlargers* is a sequence $\{G_r\}_{r=1}^{\infty}$ of graphs, where G_r is an (n_r, k, ϵ)-enlarger, k and ϵ are positive constants independent of r, and $n_r \to \infty$ with r. The term *linear* appears in the definition because the number of edges of G_r is linear as a function of its order n_r (being equal to $kn_r/2$) as r tends to infinity. A *family of linear magnifiers* is defined analogously. In view of the preceding lemma, the search for families of linear magnifiers and linear enlargers amounts to essentially the same thing. In fact, all explicit constructions of such families have so far relied on demonstrating their enlarging property.

Besides being of theoretical interest in their own right, such families are important because they can be used to construct families of another type of graph known as an *expander* graph.

Definition 1.4. *An (n, k, d)-expander is a bipartite graph G of maximum valency k on two vertex sets known as I (inputs) and O (outputs), where $|I| = |O| = n$ and, for every subset $X \subseteq I$ satisfying $|X| \leq n/2$, we have*

$$|\Gamma(X)| \geq \left(1 + d\left(1 - \frac{|X|}{n}\right)\right)|X|.$$

The graph G is called a strong expander if the above inequality holds for all $X \subseteq I$. ∎

A *family of k-regular linear expanders* with expansion d is a sequence $\{G_r\}_{r=1}^{\infty}$ of graphs, where G_r is an (n_r, k, d)-expander ($d > 0$) and $n_r \to \infty$ as $r \to \infty$. Expanders can be obtained in a simple way from magnifiers. Let G be a (not necessarily simple) graph on vertex set $V = \{v_1, \ldots, v_n\}$. The *augmented double cover* of G is the bipartite graph $\tilde{G}$ on input set $I_n = \{\, x(i) \mid 1 \leq i \leq n \,\}$ and output set $O_n = \{\, y(j) \mid 1 \leq j \leq n \,\}$. Suppose that there are r edges between v_i and v_j in G. Then, if $i \neq j$ there are r edges between $x(i)$ and $y(j)$ in $\tilde{G}$, whilst if $i = j$ there are $r + 1$ edges between $x(i)$ and $y(j)$ in $\tilde{G}$. The following result, due to Alon, illustrates how we can use this construction to make expanders from magnifiers.

Lemma 1.5. *The augmented double cover of an (n, k, i)-magnifier is an $(n, k + 1, d)$-expander with $d = i$.*

Proof. Let $G = (V, E)$ be a (n, k, i)-magnifier and $\tilde{G}$ be its augmented double cover. Let $\tilde{X} \subseteq I_n$ with $|\tilde{X}| \leq n/2$ and suppose X is the corresponding subset of V. Then, if $\tilde{\Gamma}(\tilde{X}) \subseteq O_n$ is the set of neighbours of $\tilde{X}$ in $\tilde{G}$, clearly

$$\begin{aligned}|\tilde{\Gamma}(\tilde{X})| &= |X| + |\Gamma(X) \setminus X| \\ &\geq |X| + i\,|X|\end{aligned}$$

because G is an (n,k,i)-magnifier. Hence, because

$$1+i \geq 1+i\left(1-\frac{|X|}{n}\right),$$

and $|X| = |\tilde{X}|$, then we have

$$|\tilde{\Gamma}(\tilde{X})| \geq \left(1+i\left(1-\frac{|\tilde{X}|}{n}\right)\right)|\tilde{X}|$$

as required. ∎

Thus it is clear from the above that the augmented double covers of a family of linear magnifiers will be a family of linear expanders.

Expander graphs have several applications, among them being their use in parallel sorting networks [**AKS**], in the construction of sparse graphs with dense long paths [**EGS**], and in the construction of graphs with special connectivity properties known as *superconcentrators.*

Definition 1.6. *An (n,k)-superconcentrator is a directed acyclic graph with n input vertices, n output vertices, and at most kn edges, with the following property: for every $1 \leq r \leq n$ and every two sets A of r inputs and B of r outputs there are r vertex-disjoint directed paths from the vertices of A to those of B (establishing a one-to-one correspondence between them, but not a predetermined one).* ∎

A family of linear superconcentrators of density k is a set of $(n_r, k+o(1))$-superconcentrators $(r = 1, 2, \ldots)$ with $n_r \to \infty$ as $r \to \infty$. That they may be derived from families of linear expanders is evinced in the following result.

Lemma 1.7. *A family of $(n_r, k, 2/(p-1))$-strong expanders, where p is a fixed integer greater than one, may be used to construct a family of linear superconcentrators of density $(2k+3)p+1$.*

Proof. [**GG**, Theorem 3]. ∎

A general criterion for good expanding families of graphs is to keep the valency k small whilst making the expansion d as large as possible. Hence, in view of the expression for the density in Lemma 1.7., a useful measure of quality of a family of linear expanders is the density of the derived family of linear superconcentrators; the lower the density, the better the expanders. In the next section, where we briefly review known constructions of families of expander graphs, we will give the corresponding density of the superconcentrators obtained with this in mind.

The study of superconcentrators is relevant to switching theory, where networks are needed to connect many users with a minimum number of switches employed [**P2**]. They are also useful in theoretical computer science, in areas such as pebbling ([**P1**], [**Va**]). Unfortunately, the explicit construction of families of linear superconcentrators is quite difficult, and so far all constructions have used families of linear expanders. It is known by nonconstructive counting arguments that there exist families of linear superconcentrators of density 36 [**Ba**], but so far no explicit construction has attained this value.

1.2 Known Expander Constructions

Margulis [**Ma**] constructed the first explicit family of linear expanders using group representation theory, but was unable to give a value for the coefficient of expansion d. His expanding graphs were defined by labelling the vertices of each bipartite block with the elements of $\mathbf{Z}_m \times \mathbf{Z}_m$ (where $\mathbf{Z}_m$ is the ring of integers modulo m) and joining a typical vertex (x, y) in the first block to vertices in the second block of the form (X, Y), where X and Y are certain affine linear functions of x and y. Gabber and Galil [**GG**] adopted a very similar construction but were able to supply an explicit value for d. Hence, they were able to show that their strong expanders gave superconcentrators of density 273. By improving on these methods Jimbo and Maruoka [**JM**] constructed a family of linear (strong) expanders which yielded superconcentrators of density 248, but so far much the best result has been obtained by Lubotzky, Phillips and Sarnak [**LPS**]. Their enlargers, which are Cayley graphs of $PGL(2, \mathbf{Z}_q)$, give superconcentrators of density 58 (using the analysis in [**Bu**]).

1.3 Summary of New Results

There follows a brief outline of new results contained in this thesis. In Chapter 2 we present some upper bounds on $i(G)$ in terms of elementary graph properties such as order, valency, and the number of cycles of length 3 and 4, while Chapter 3 is primarily devoted to the relationship between $i(G)$ and the genus of G, in particular the fact that families of linear magnifiers must have 'large' genus compared to their order. We also show that a class of graphs known as finite element graphs never supply families of linear magnifiers. Chapter 4 is divided into two sections, the first of which establishes results concerning the magnifying and enlarging properties of Hamiltonian cubic graphs, while the second section introduces a class of graphs known as Shift graphs and derives a general formula for the spectrum of such a graph. In Chapter 5 a technical result is derived from Klawe's theorem on non-expanding families of graphs and applied to the formula obtained in Chapter 4 to show that no families of shift graphs ever form families of linear enlargers. Chapter 6 consists of further consequences of Klawe's theorem, in particular proofs that several important classes of Cayley graphs do not form families of linear enlargers. In Chapter 7 we present an upper bound for the 'gap' $\epsilon = k - \lambda_1(G)$ of a general vertex-transitive graph G using group representation theory, and in Chapter 8 we discuss the problem of finding Ramanujan graphs and give several possible criteria for establishing that a graph is Ramanujan. Finally, in Chapter 9 we use the methods of Buck to derive an infinite number of families of expander graphs with expansion converging to one.

Chapter Two

Elementary Results on i(G)

In this chapter we establish some elementary results on the isoperimetric number $i(G)$ of a k-regular graph G. We give a general upper bound for $i(G)$, and also relate it to the number of cycles of length 3 and 4 in G. The graph G is assumed to be simple and connected throughout.

2.1 An Upper Bound for $i(G)$

Theorem 2.1. *Let $G = (V, E)$ be a k-regular graph of order n. Then*

$$i(G) \leq \frac{\lceil n/2 \rceil}{\lfloor n/2 \rfloor} \left(1 - \prod_{r=1}^{k} \left(\frac{\lceil n/2 \rceil - r}{n - r} \right) \right),$$

where $\lceil x \rceil$, $\lfloor x \rfloor$ denote the smallest integer greater than or equal to x, and the largest integer less than or equal to x, respectively.

Proof. Consider the set H of all sets of $\lfloor n/2 \rfloor$ vertices from V. For any $X \in H$ denote by $b(X)$ the quantity $|\Gamma(X) \setminus X|$, which is the number of vertices of G adjacent to the set X but not in X itself. Then $b(X) = \lfloor n/2 \rfloor i(X)$. The average value of the function $b(X)$ on H is

$$B(G) = \sum_{X \in H} b(X) \Big/ \binom{n}{\lfloor n/2 \rfloor}. \tag{2.1}$$

We evaluate the above sum by determining for how many sets $X \in H$ a general vertex v appears as a member of $\Gamma(X) \setminus X$.

Suppose the neighbours of the vertex $v \in V$ are $N(v) = \{w_1, \ldots, w_k\}$. Then, for each $X \in H$ for which $v \in \Gamma(X) \setminus X$, there is an integer r between 1 and k such that exactly r of the neighbours of v belong to X. The remaining $\lfloor n/2 \rfloor - r$ vertices of X must come from the set $V \setminus \{v, w_1, \ldots, w_k\}$. Thus

$$\#\{\, X \in H \mid v \notin X \quad \text{and} \quad |X \cap N(v)| = r \,\} = \binom{k}{r}\binom{n-k-1}{\lfloor n/2 \rfloor - r}.$$

Thus, since this quantity is independent of the vertex v, we have

$$\sum_{X \in H} b(X) = n \sum_{r=1}^{k} \binom{k}{r}\binom{n-k-1}{\lfloor n/2 \rfloor - r}. \tag{2.2}$$

Now it is clear that

$$\begin{aligned}
\sum_{r=0}^{k} \binom{k}{r}\binom{n-k-1}{\lfloor n/2 \rfloor - r} &= \text{ the coefficient of } \quad x^{\lfloor n/2 \rfloor} \quad \text{in} \quad (1+x)^k (1+x)^{n-k-1} \\
&= \text{ the coefficient of } \quad x^{\lfloor n/2 \rfloor} \quad \text{in} \quad (1+x)^{n-1} \\
&= \binom{n-1}{\lfloor n/2 \rfloor}.
\end{aligned}$$

Thus we have

$$\sum_{r=1}^{k} \binom{k}{r}\binom{n-k-1}{\lfloor n/2 \rfloor - r} = \binom{n-1}{\lfloor n/2 \rfloor} - \binom{n-k-1}{\lfloor n/2 \rfloor} \tag{2.3}$$

and, together with (2.1) and (2.2), this implies that

$$B(G) = n\left(\binom{n-1}{\lfloor n/2 \rfloor} - \binom{n-k-1}{\lfloor n/2 \rfloor}\right) \Big/ \binom{n}{\lfloor n/2 \rfloor}$$

Hence, since the average value of i on the members of H is

$$\bar{\imath} = B(G)/\lfloor n/2 \rfloor$$

then there must exist an $X \in H$ for which $i(X) \le \bar{\imath}$. Furthermore, $|X| \le n/2$, so that $i(G) \le i(X) \le \bar{\imath}$ and we have

$$i(G) \le n\left(\binom{n-1}{\lfloor n/2 \rfloor} - \binom{n-k-1}{\lfloor n/2 \rfloor}\right) \Big/ \lfloor n/2 \rfloor \binom{n}{\lfloor n/2 \rfloor}$$

which, upon simplifying, leads to the required result. ∎

Corollary 2.2. *Let $\{G_r\}_{r=1}^{\infty}$ be a family of k-regular graphs, with the order of G_r tending to infinity with r. Then*

$$\limsup_{r\to\infty} i(G_r) \leq 1 - 2^{-k}.$$

Proof. Clearly the bound on $i(G)$ in Theorem 2.1 tends to $1 - 2^{-k}$ as the order of G tends to infinity. ∎

2.2 $i(G)$ and Girth

There are several known results connecting the magnifying and enlarging properties of a graph with structural properties such as the diameter of the graph. For example, Alon and Milman [**AM**,Theorem 2.7] have shown that the diameter of a family of linear enlargers must be $O(\log n)$ (where n is the number of vertices) as n tends to infinity. The following result shows that, for graphs of low edge-connectivity to form magnifying families, the girth of the graphs must be bounded.

Theorem 2.3. *Let $\{G_r\}_{r=1}^{\infty}$ be a family of k-regular graphs $(k \geq 3)$ with n_r, the order of G_r, tending to infinity with r. Suppose that the edge-connectivity of each graph is at most $k-1$. Then either $\liminf_{r\to\infty} i(G_r) = 0$ or there is a constant M such that the girth of G_r is at most M for all r.*

Proof. Let G_r be λ-edge-connected for some $1 \leq \lambda \leq k-1$ (the lower bound is to ensure that the graph is connected). Then there exists a set E_r of λ edges of G_r, and a partition $V = A_r \cup B_r$ of the vertices of G_r with the properties:

(1) $A_r \neq \emptyset$ and $B_r \neq \emptyset$;

(2) The set of edges with one vertex in A_r and the other in B_r is precisely E_r.

Without loss of generality we may suppose that $|A_r| \leq |B_r|$ so that $|A_r| \leq n_r/2$. Then

$$\begin{aligned} i(G_r) &\leq i(A_r) \\ &\leq (k-1)/|A_r|. \end{aligned}$$

Hence either $\liminf_{r\to\infty} i(G_r) = 0$, or $|A_r|$ must be bounded for all r, say by M. In the latter case, since the graph G_r is k-regular, the vertex subgraph of G_r on A_r has

$(k|A_r| - \lambda)/2$ edges, and

$$\frac{(k|A_r| - \lambda)}{2} \geq \frac{k(|A_r| - 1) + 1}{2}$$
$$\geq \frac{3|A_r| - 2}{2} \quad \text{if } k \geq 3,$$

which is at least $|A_r|$, because $\lambda < k$ implies that $|A_r| \geq 2$. Hence this subgraph contains a cycle, of length necessarily no more than M. Thus the girth of G_r is at most M, which completes the proof. ∎

2.3 Cubic Graphs, Four-Cycles and $i(G)$

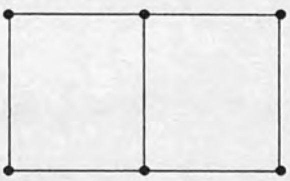
In this section we concentrate on cubic (that is, 3-regular) graphs and relate $i(G)$ to the number of cycles of length 4 in the graph G. Given a cubic graph $G = (V, E)$ of order n, we define

$$s(G) = \#\{\,\{a,b,c,d\} \subseteq V \mid \quad \text{the vertex subgraph of } G \text{ on } \{a,b,c,d\} \text{ is a 4-cycle}\,\}.$$

We also define

$$t(G) = \#\{\,\{a,b,c,d,e,f\} \subseteq V \mid \quad \text{the vertex subgraph of } G \text{ on } \{a,b,c,d,e,f\} \cong H\,\}$$

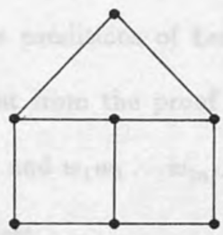
where H is the graph shown below.



We define a related graph $G_1 = (V_1, E_1)$ as follows: To each 4-set counted in $s(G)$ there corresponds a vertex in V_1, and two vertices of G_1 are adjacent if and only if the vertices of the two corresponding 4-cycles of G together form a vertex subgraph of G that is isomorphic to H. From this definition it follows at once that we have $|V_1| = s(G)$ and $|E_1| = t(G)$.

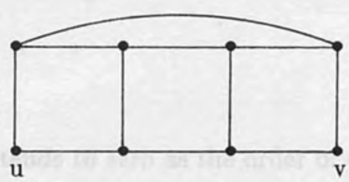
Lemma 2.4. *(i) Suppose the cubic graph G is 3-edge-connected and has order greater than 8. Then G_1 has maximum vertex degree at most 2.*

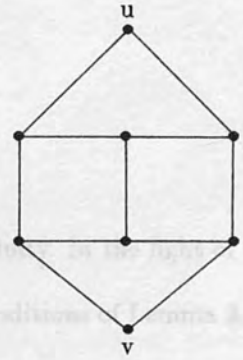
(ii) If, additionally, G contains no vertex subgraph isomorphic to the graph F (where F is shown below)



then G_1 is either acyclic or is a graph which is just a single cycle.

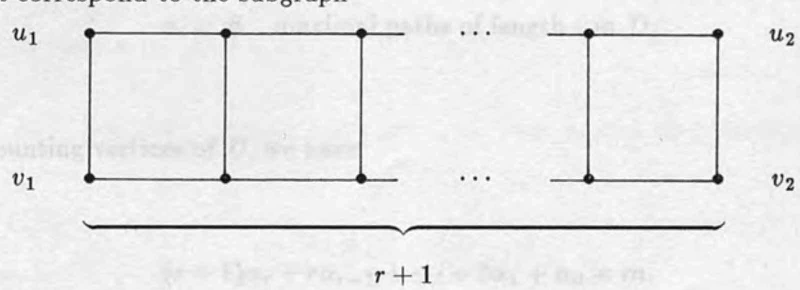
Proof. (i) Suppose that the conditions are satisfied, but G_1 contains a vertex with three neighbours. Then it is easy to see that G contains one of the subgraphs below.





If $\{u, v\} \in E$ then G, being connected, has order 8, whilst if $\{u, v\} \notin E$ then G is 2-edge-connected, which in each case is a contradiction. Hence G_1 has maximum vertex degree at most two.

(ii) If G contains no subgraph isomorphic to F and also satisfies the conditions of part (i) then clearly G_1 has no 3-cycles, and it is then easy to see that a path of length r in G_1 must correspond to the subgraph



of G. If the end vertices of the path in G_1 are adjacent then we have $\{u_1, v_1\} = \{u_2, v_2\}$, in which case G (being connected) will be equal to the above subgraph in which the edges $\{u_1, v_1\}$ and $\{u_2, v_2\}$ have been identified with each other. Then G_1 will simply consist of a single cycle of length $r+1$, as required. ∎

Suppose we have satisfied the conditions of Lemma 2.4, and that G_1 consists of a single cycle. Then, as is evident from the proof of lemma 2.4(ii), G contains as a subgraph two paths $x_1x_2 \ldots x_{\lfloor n/4 \rfloor}$ and $w_1w_2 \ldots w_{\lfloor n/4 \rfloor}$, together with the edges $\{x_i, w_i\}$ for $i = 1, 2, \ldots, \lfloor n/4 \rfloor$. Then, if we set

$$X = \{x_1, w_1, x_2, w_2, \ldots, x_{\lfloor n/4 \rfloor}, w_{\lfloor n/4 \rfloor}\},$$

we have

$$i(G) \leq i(X)$$

$$\leq \frac{4}{2\lfloor n/4 \rfloor}.$$

Hence, $i(G)$ tends to zero as the order of G tends to infinity. In the light of what follows, we shall call a cubic graph G *proper* if it satisfies the conditions of Lemma 2.4 and is such that G_1 is acyclic.

Lemma 2.5. *Let D be an acyclic graph of order m with maximum vertex degree at most 2. If D has at least $1 + \frac{mr}{r+1}$ edges then it contains a path of length at least $r+1$.*

Proof. Suppose D has no path of length at least $r+1$. The maximal paths of D are precisely its connected components, so that if

$$\alpha_i = \# \quad \text{maximal paths of length } i \text{ in } D,$$

then, on counting vertices of D, we have

$$(r+1)\alpha_r + r\alpha_{r-1} + \cdots + 2\alpha_1 + \alpha_0 = m. \tag{2.4}$$

Similarly, by counting edges in D, we obtain

$$r\alpha_r + (r-1)\alpha_{r-1} + \cdots + \alpha_1 \geq 1 + \frac{mr}{r+1}. \tag{2.5}$$

Subtracting (2.5) from (2.4) gives

$$\begin{aligned} \alpha_r + \cdots + \alpha_0 &\leq m - \left(\frac{mr}{r+1}\right) - 1 \\ &= \frac{m}{r+1} - 1. \end{aligned} \tag{2.6}$$

Dividing equation (2.5) by r implies that

$$\alpha_r + \cdots + \alpha_1 + \alpha_0 \geq \frac{m}{r+1} + \frac{1}{r}. \tag{2.7}$$

Hence, by combining (2.6) and (2.7) we see that

$$\frac{m}{r+1} - 1 \geq \frac{m}{r+1} + \frac{1}{r}$$

which is clearly absurd for positive r, so that by contradiction there must exist a path in D of length at least $r+1$. ∎

In fact, it is easy to see that the above result is the best possible: For, if $D = (V, E)$ is the union of α disjoint r-paths then $|V| = \alpha(r+1)$ and

$$\begin{aligned} |E| &= \alpha r \\ &= \frac{\alpha(r+1)r}{(r+1)} \end{aligned}$$

and D has no path of length $r+1$.

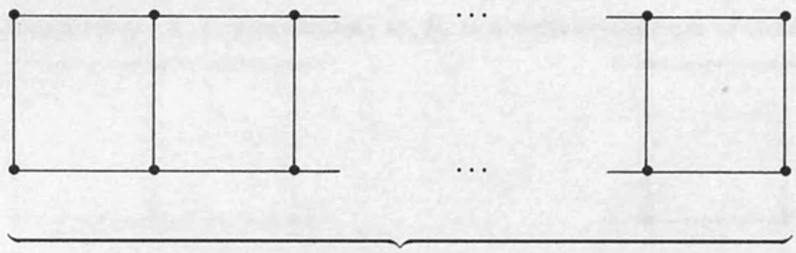
Corollary 2.6. *Let G be a proper cubic graph of order n, and suppose that*

$$t(G) \geq \left(\frac{r}{r+1}\right) s(G) + 1.$$

Then

$$i(G) \leq \max\left\{\frac{2}{\lfloor n/4 \rfloor}, \frac{2}{r+3}\right\}.$$

Proof. If $t(G) \geq (\frac{r}{r+1})s(G) + 1$ then, by Lemma 2.5, the graph G_1 has a path of length at least $r+1$. This corresponds in G to the following vertex subgraph



$r+2$

This subgraph has $2r+6$ vertices and, if we denote the set of these vertices by X, then in the graph G we have

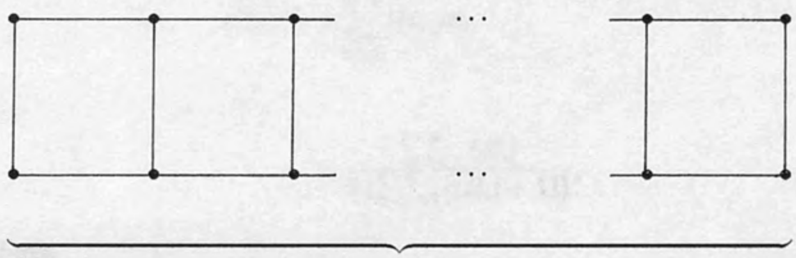
$$|\Gamma(X) \setminus X| \leq 4.$$

Thus, if $2r+6 \leq n/2$ then it follows that

$$i(G) \leq i(X)$$

$$\leq \frac{4}{2r+6}.$$

On the other hand, if $2r+6 > n/2$ then we choose the subgraph



$\lfloor n/4 \rfloor - 1$

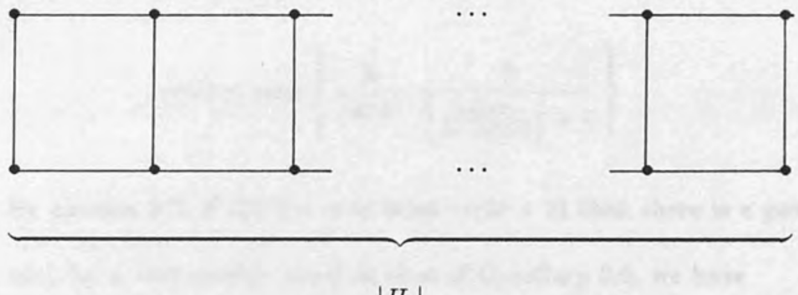
which gives

$$i(G) \leq \frac{2}{\lfloor n/4 \rfloor}$$

and this completes the proof. ■

Lemma 2.7. *Let G be a proper cubic graph of order n. Further, suppose that no two four-cycles of G have more than one edge in common. Then if $s(G)/n \geq r/(2r+2)$, then G_1 contains a path of length $r-1$.*

Proof. Let $H_1, \ldots, H_\alpha$ be the components of G_1, so that H_i is a path with $|H_i|$ vertices, of length $|H_i| - 1$. Corresponding to H_i is a vertex subgraph of G of the form



because G is proper. Also, because no two four-cycles of G have more than one common edge, if X_i is the set of vertices in the above subgraph of G corresponding to H_i, then $X_i \cap X_j = \emptyset$ if $i \neq j$. (For otherwise either H_i and H_j are connected together in G_1, or there exist two four-cycles in G with at least two common edges, which is a contradiction). Hence, since $|X_i| = 2(|H_i| + 1)$, we have

$$n \geq 2 \sum_{i=1}^{\alpha} (|H_i| + 1).$$

Thus

$$s(G)/n = \sum_{i=1}^{\alpha} |H_i|/n$$

$$\leq \frac{\sum_{i=1}^{\alpha} |H_i|}{2\left(\sum_{i=1}^{\alpha} (|H_i| + 1)\right)},$$

so that we have

$$s(G)/n \leq \max_{1 \leq i \leq \alpha} \frac{|H_i|}{2(|H_i| + 1)}.$$

Hence, if $s(G)/n \geq r/(2r+2)$, then there is an i for which

$$\frac{|H_i|}{2(|H_i| + 1)} \geq \frac{r}{2(r+1)},$$

from which we see that $|H_i| \geq r$, as required. ∎

The results of Corollary 2.6 and Lemma 2.7 provide an upper bound for $i(G)$ in terms of the parameter $s(G)$, as we now show.

Theorem 2.8. *Let G be a proper cubic graph of order n, with the property that no two four-cycles have more than one edge in common. Then*

$$i(G) \leq \max\left\{\frac{2}{\lfloor n/4 \rfloor}, \frac{2}{\left\lfloor \frac{2s(G)}{n-2s(G)} \right\rfloor + 1}\right\}.$$

Proof. By Lemma 2.7, if $s(G)/n$ is at least $r/(2r+2)$ then there is a path of length $r-1$ in G_1 and, by a very similar proof to that of Corollary 2.6, we have

$$i(G) \leq \max\left\{\frac{2}{\lfloor n/4 \rfloor}, \frac{2}{r+1}\right\}.$$

Now $s(G)/n \leq r/(2r+2)$ if and only if $r \leq 2s(G)/(n-2s(G))$ and, since r is an integer, we may choose

$$r = \left\lfloor \frac{2s(G)}{n-2s(G)} \right\rfloor,$$

which gives the required result. ∎

Corollary 2.9. *Let $\{G_r\}_{r=1}^{\infty}$ be a family of proper cubic graphs, where G_r has order n_r which tends to infinity with r. Then, if no two four-cycles of G_r have more than one common edge, and $s(G_r)/n_r \to \frac{1}{2}$ as $r \to \infty$, then $i(G_r) \to 0$ as $r \to \infty$.*

Proof. If $s(G_r)/n_r \to \frac{1}{2}$ then

$$\left\lfloor \frac{2s(G_r)}{n_r - 2s(G_r)} \right\rfloor \to \infty$$

as $r \to \infty$. The result now follows from that of Theorem 2.8. ∎

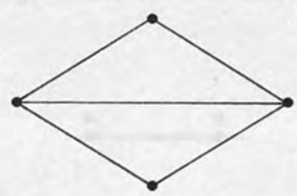
In fact, it is clear that for any proper cubic graph G of order n in which no two four-cycles have more than one common edge we have $s(G)/n < 1/2$, since equality would imply that G_1 was a single cycle.

2.4 Cubic Graphs, Three-Cycles and $i(G)$

In this section we examine the relationship between the number of three-cycles in a cubic graph G and its isoperimetric number. We assume throughout that $G = (V, E)$ is a cubic graph of order n that is not isomorphic to the complete graph K_4 on four vertices. We define

$$\gamma(G) = \#\{\,\{a,b,c,d\} \subseteq V \mid \quad \text{the vertex subgraph of } G \text{ on } \{a,b,c,d\} \cong L\,\}$$

where L is the graph shown below.



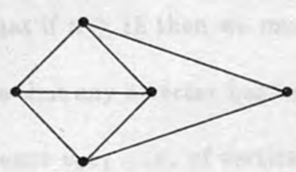
Theorem 2.10. *(a)* $\gamma > n/6$ *and* $n \geq 18 \Longrightarrow i(G) \leq 1/3$.

(b) $\gamma > n/5$ *and* $n \geq 28 \Longrightarrow i(G) \leq 2/7$.

(c) $\gamma > 3n/14$ *and* $n \geq 24 \Longrightarrow i(G) \leq 1/6$.

Proof. We form the related graph $G_2 = (V_2, E_2)$ as follows: For each 4-set counted in $\gamma(G)$ there corresponds a vertex in V_2 of degree two, which we shall call an *a-vertex*, and for each vertex of G not in such a 4-set there is a vertex in V_2 of degree three, which we shall call a *b-vertex*. Two vertices $u, v \in V_2$ are adjacent in G_2 if and only if the corresponding vertices and/or subgraphs were adjacent in G, and with the same multiplicity. Hence G_2 contains γ a-vertices and $(n - 4\gamma)$ b-vertices. We shall always assume that G is connected, in which case it is easy to see that G_2 is too.

(a) If there exists an a-vertex and a b-vertex with two edges between them then G contains the subgraph X shown below.

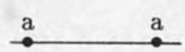


Clearly $i(X) \leq \frac{1}{5}$ so that $n \geq 10$ implies that $i(G) \leq \frac{1}{5}$. Hence, if $n \geq 18$ then we may assume that each a-vertex has two *distinct* neighbours.

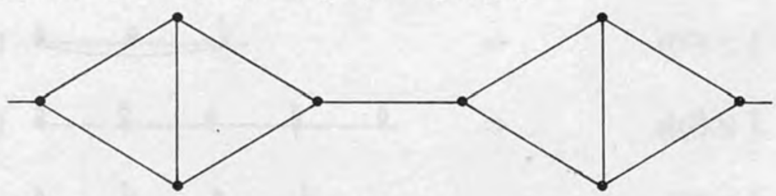
Suppose that in G_2 any two a-vertices are at a distance at least three apart. Then we may assign (uniquely) two b-vertices to each a-vertex (namely its neighbours in G_2) so that no b-vertex is assigned to more than one a-vertex. Hence, by counting b-vertices we have

$$n - 4\gamma \geq 2\gamma$$

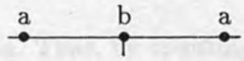
so that $\gamma \leq n/6$. Thus, if $\gamma > n/6$ then there must exist two a-vertices which are at distance at most two apart in G_2. There are two cases to consider: If G_2 contains the subgraph



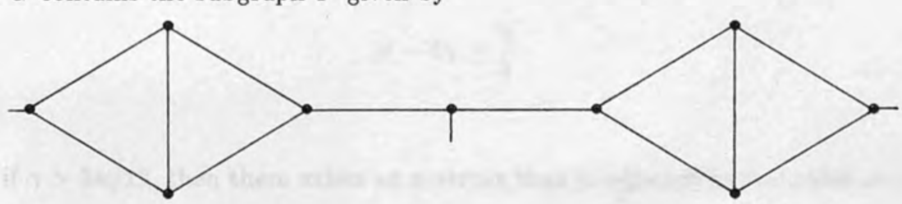
then it follows that G contains the subgraph X given by



and clearly $i(X) \leq 2/8$, so that if $n \geq 16$ (this is to ensure that $|X| \leq n/2$ so that $i(G) \leq i(X)$ holds) then we deduce that $i(G) \leq 1/4$. Otherwise G_2 must contain the subgraph



so that G contains the subgraph Y given by



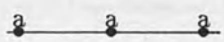
which satisfies $i(Y) \leq 3/9$, so that if $n \geq 18$ then we must have $i(G) \leq 1/3$.

(b) As in (a) we may assume that any a-vertex has two distinct neighbours. A *reduced walk* of length r in G_2 is a sequence $v_0 v_1 \ldots v_r$ of vertices in V_2 such that $\{v_i, v_{i+1}\} \in E_2$

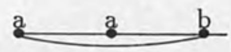
for $0 \le i \le r-1$ and $v_i \ne v_{i+2}$ for $0 \le i \le r-2$. Suppose that every a-vertex has at most one reduced walk of length one or two to other a-vertices. Then it is easy to see that each a-vertex has a neighbouring b-vertex that is adjacent to no other a-vertex. Hence, counting b-vertices gives $(n - 4\gamma) \ge \gamma$, so that, if $\gamma > n/5$ then there must exist an a-vertex with two reduced walks of length one or two to other a-vertices. This implies that one of the five configurations shown below must exist in G_2. For each configuration X we give the corresponding upper bound for $i(\tilde{X})$ (where $\tilde{X}$ is the subgraph of G that yielded the congifuration X in G_2). Then, provided the order n of G satisfies $n \ge 2|\tilde{X}|$, we have $i(G) \le i(\tilde{X})$. (In fact it is easy to see, by considering the orders of each of the corresponding subgraphs of G, that $n \ge 28$ is sufficient).

(i) 

$\Rightarrow$ $i(G) \le \frac{3}{13}$

(ii) 

$\Rightarrow$ $i(G) \le \frac{1}{6}$

(iii) 

$\Rightarrow$ $i(G) \le \frac{1}{9}$

(iv) 

$\Rightarrow$ $i(G) \le \frac{2}{7}$

(v) 

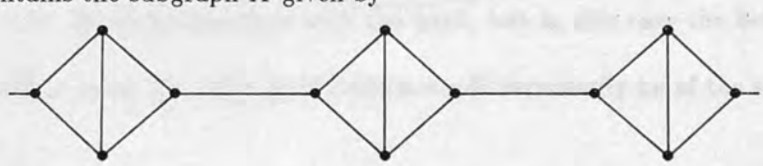
$\Rightarrow$ $i(G) \le \frac{1}{5}$

The largest upper bound in the five cases for $i(G)$ is $2/7$, from which the result follows.

(c) Suppose that every a-vertex has at least one neighbour which is a b-vertex. We may then assign a b-vertex in this way to each a-vertex so that each b-vertex can be so assigned to at most three a-vertices. Thus, by counting b-vertices

$$n - 4\gamma \ge \frac{\gamma}{3}.$$

Hence, if $\gamma > 3n/13$, then there exists an a-vertex that is adjacent to two other a-vertices, so that G contains the subgraph X given by

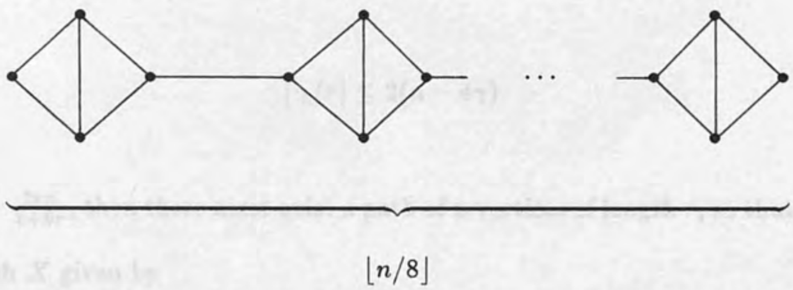


which satisfies $i(X) \leq 2/12$. Thus, if $n \geq 24$, then we have $i(G) \leq i(X) \leq 1/6$, as required. ∎

Theorem 2.11. *If G is a cubic graph of order n, then*

$$i(G) \leq \max\left\{ \frac{2}{\lfloor n/8 \rfloor}, \frac{1}{2\left\lfloor \frac{\gamma}{2(n-4\gamma)} \right\rfloor} \right\}$$

Proof. Firstly, suppose that G_2 consists of a single cycle. Then we let X be the subgraph of G given by



so that $i(G) \leq i(X) \leq \frac{2}{\lfloor n/8 \rfloor}$. From now on we assume that G_2 is not a single cycle. Suppose that, in G_2, every a-vertex lies on a path (composed entirely of a-vertices) of length at most $r-1$. Since G_2 is not a cycle, there are two possible cases.

(a) The path in G_2 is of the form

and we associate the two b-vertices at the ends of this path with the path. Since each b-vertex has valency three, it can be associated with a path in this manner at most three times.

(b) The path is of the form

and we associate the b-vertex above with the path, but in this case the b-vertex can be associated with at most one other path (which would necessarily be of the type indicated in case (a)).

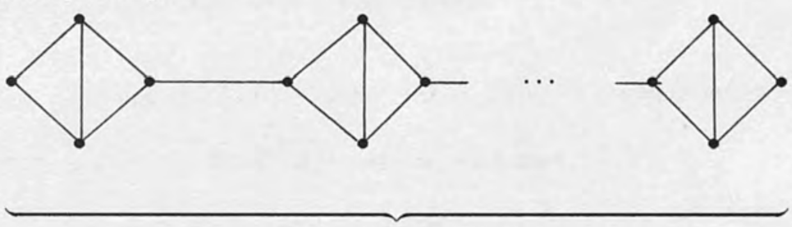
Suppose there are α paths of the type shown in (a), and β of the type shown in (b). Then, on counting the number of b-vertices in the manner suggested above, we have

$$\frac{2\alpha}{3} + \frac{\beta}{2} \leq n - 4\gamma.$$

Hence $(\alpha + \beta) \leq 2(n - 4\gamma)$. Now $\alpha + \beta$ is the total number of maximal paths containing only a-vertices in the graph G_2. If all such paths have length less than or equal to $r - 1$, then the number of them is clearly minimised when as many of them as possible have length $r - 1$. In other words we have $\lceil \gamma/r \rceil \leq \alpha + \beta$, so that

$$\lceil \gamma/r \rceil \leq 2(n - 4\gamma).$$

Thus, if $\gamma > \frac{2rn}{1+8r}$, then there must exist a path of a-vertices of length r, so that G contains the subgraph X given by



$r + 1$

and it is clear that $i(X) \leq \frac{1}{2(r+1)}$, from which we can obviously deduce that

$$i(G) \leq \max\left\{\frac{2}{\lfloor n/8 \rfloor}, \frac{1}{2(r+1)}\right\}.$$

Now $\gamma > \frac{2rn}{1+8r}$ if and only if $r < \frac{\gamma}{2(n-4\gamma)}$, so we choose

$$r = \left\lfloor \frac{\gamma}{2(n - 4\gamma)} \right\rfloor - 1,$$

from which the required result follows. ∎

Finally, the above result implies an asymptotic result for the value of $i(G)$ as the value $\gamma/n \to \frac{1}{4}$.

Corollary 2.12. *Let $\{G_r\}_{r=1}^{\infty}$ be a family of cubic graphs of order n_r that tends to infinity with r. Then, if $\gamma(G_r)/n_r \to \frac{1}{4}$ as $r \to \infty$, then $i(G) \to 0$.*

Proof. Use the upper bound for $i(G)$ in Theorem 2.11. ∎

In fact, if G is a cubic graph of order n then it is easy to see that $\gamma(G)/n \leq \frac{1}{4}$ with equality if and only if G_2 consists of a single cycle of a-vertices.

Chapter Three

Isoperimetric Number and Genus

In this chapter we derive upper bounds for $i(G)$, where G is any graph whose genus is sufficiently small compared to its order. (For the definition of the genus of a graph see [**Wi**, p.69]). We apply these results to determine some results concerning the average face size of families of linear magnifiers. Finally we look at graphs whose average face size is an integer, and at a class of graphs known as Finite Element Graphs.

3.1 Bounding $i(G)$ by the Genus

For this section the following result is fundamental.

Lemma 3.1. *(a) Suppose G is a planar (that is, genus 0) graph of order n. Then there exists a partition $V = A \cup B \cup C$ of its vertices such that:*

(i) No edge of G joins a vertex of A to one of B;

(ii) $|A|, |B| \leq 2n/3$;

(iii) $|C| \leq 2\sqrt{2n}$.

(b) Suppose G is a graph of genus g and order n. Then there exists a subset X of the vertices of G such that:

(i) $|X| \leq \sqrt{2n}$;

(ii) The genus of the vertex subgraph $G \setminus X$ is at most $g - 1$.

Proof. (a) Lipton and Tarjan [**LT**, Corollary 2].

(b) Sider [**Si**]. ∎

From Lemma 3.1(a) we may obtain a simple upper bound for the isoperimetric number $i(G)$ of a planar graph G.

Theorem 3.2. *Let G be a planar graph of order $n > 72$. Then*

$$i(G) \leq \left(\sqrt{\frac{n}{72}} - 1\right)^{-1}.$$

Proof. Let A, B, C be as given in Lemma 3.1(a). Then we may suppose, without loss of generality, that $|A| \leq |B|$, whence it follows that $|A| \leq n/2$. We also have

$$|A| + |B| \geq n - 2\sqrt{2n}$$

and

$$|B| \leq 2n/3.$$

We thus deduce that

$$\begin{aligned} |A| &\geq n/3 - 2\sqrt{2n} \\ &> 0 \qquad (\text{if } n > 72). \end{aligned}$$

Now it is clear from condition (i) of Lemma 3.1(a) that $\Gamma(A) \setminus A \subseteq C$, so that

$$\begin{aligned} i(A) &\leq \frac{|C|}{|A|} \\ &\leq \frac{2\sqrt{2n}}{(n/3) - 2\sqrt{2n}} \\ &= \left(\sqrt{\frac{n}{72}} - 1\right)^{-1}. \end{aligned}$$

Now, because $|A| \leq n/2$, we have $i(G) \leq i(A)$, from which the theorem follows immediately. ∎

As a simple corollary to the above, we have the following result.

Corollary 3.3. *There exist no families of linear magnifiers (and hence enlargers) of planar graphs.*

Proof. If $\{G_r\}_{r=1}^{\infty}$ is a family of k-regular planar graphs such that the order n_r of G_r tends to infinity with r, then by Theorem 3.2

$$i(G_r) \le \left(\sqrt{\frac{n_r}{72}} - 1\right)^{-1}$$

if $n_r > 72$. Hence $i(G_r) \to 0$ as $r \to \infty$, so that $\{G_r\}_{r=1}^{\infty}$ cannot be a family of linear enlargers. ∎

We now make use of Lemma 3.1(b) to extend Theorem 3.2 to graphs of higher genus.

Theorem 3.4. *If G is a graph of order n and has genus g which satisfies $g < \frac{1}{3}\sqrt{\frac{n}{2}} - 2$, then*

$$i(G) \le \left(\sqrt{\frac{n}{2}}\left(\frac{1}{3(g+2)}\right) - 1\right)^{-1}.$$

Proof. By Lemma 3.1(b) there exists a subset X_1 of the vertices of G such that $|X_1| \le \sqrt{2n}$ and the vertex subgraph $G \setminus X_1$ has genus at most $g-1$. Similarly, applying the same lemma to the graph $G \setminus X_1$ implies the existence of a subset X_2 of vertices of $G \setminus X_1$ with $|X_2| \le \sqrt{2n}$ and such that the vertex subgraph $G \setminus (X_1 \cup X_2)$ has genus at most $g-2$. Continuing in this way it is clear that there is a subset $X = X_1 \cup \ldots \cup X_g$ of the vertices of G with $|X| \le g\sqrt{2n}$ and such that the vertex subgraph $H = G \setminus X$ is planar.

By Lemma 3.1(a) there exists a partition $A \cup B \cup C$ of the vertices of H such that (since H clearly has order at most n) the conditions (i), (ii) and (iii) of Lemma 3.1(a) apply. As before, we assume that $|A| \le |B|$ so that $|A|$ is at most half the order of H, which is itself at least $n - g\sqrt{2n}$. We then have

$$|A| + |B| \ge (n - g\sqrt{2n}) - 2\sqrt{2n}$$

which, together with the fact that $|B| \le 2n/3$, implies that

$$|A| \ge \frac{n}{3} - (g+2)\sqrt{2n},$$

which is positive if $g < \frac{1}{3}\sqrt{\frac{n}{2}} - 2$.

Let $\Gamma_G(A)$ denote the set of vertices of G adjacent to some vertex of A, and define $\Gamma_H(A)$ analagously. Suppose that $v \in \Gamma_G(A) \setminus A$. Then it is clear that either $v \in X$ or $v \in \Gamma_H(A) \setminus A$. However $\Gamma_H(A) \setminus A \subseteq C$ and clearly $X \cap C = \emptyset$, so that

$$|\Gamma_G(A) \setminus A| \leq |C| + |X|$$

$$\leq 2\sqrt{2n} + g\sqrt{2n}.$$

We deduce that, with respect to the graph G,

$$i(A) \leq \frac{(g+2)\sqrt{2n}}{(n/3) - (g+2)\sqrt{2n}},$$

and, because $|A|$ is clearly at most $n/2$, then $i(G) \leq i(A)$ and the result follows. ∎

Corollary 3.5. *Let $\{G_r\}_{r=1}^{\infty}$ be a family of k-regular graphs, such that the order of G_r is n_r and the genus is g_r. Then, if $n_r \to \infty$ with r and $g_r = o(\sqrt{n_r})$ (so that $g_r/\sqrt{n_r}$ tends to zero as r tends to infinity), we have $i(G_r) \to 0$ as $r \to \infty$.*

Proof. If $g_r = o(\sqrt{n_r})$ then, for large enough r, we have $g_r < \frac{1}{3}\sqrt{\frac{n_r}{2}} - 2$ so that, by Theorem 3.4,

$$i(G_r) \leq \left(\sqrt{\frac{n_r}{2}}\left(\frac{1}{3(g_r+2)}\right) - 1\right)^{-1}.$$

Now, as $r \to \infty$, so does $\sqrt{n_r}/g_r$, so that $i(G_r) \to 0$, as required. ∎

To summarise Corollary 3.5, a family of linear magnifiers (or enlargers) must have genus $g_r = \Omega(\sqrt{n_r})$ (that is, there exists a constant C such that $g_r \geq C\sqrt{n_r}$ for all r). Note, however, that the usefulness of this result is restricted by the fact that, if G is a graph of valency at least 7, then it has genus $g \geq n/12$, where n is the order of G ([**Wi**, Corollary 14C]). However, it remains a non-trivial result for graphs whose valency is at most 6.

3.2 Average Face Size and Isoperimetric Number

Let G be a connected k-regular graph of order n and genus g. Then the number of faces f in any embedding of G on a surface of genus g is given by Euler's formula:

$$n - \frac{kn}{2} + f = 2 - 2g. \tag{3.1}$$

(See, for example, Wilson [**Wi**, pp.70–71].)

We define the *average face size* of G to be

$$\phi(G) = \frac{kn}{f}.$$

Let $\{G_r\}_{r=1}^{\infty}$ be a family of k-regular graphs, and let $\phi_r = \phi(G_r)$. In this section we derive some conditions on the sequence $\{\phi_r\}_{r=1}^{\infty}$ if $\{G_r\}_{r=1}^{\infty}$ is a family of linear magnifiers.

Lemma 3.6. *Let $\{G_r\}_{r=1}^{\infty}$ be a family of k-regular graphs ($k \le 6$) with ϕ_r the average face size of G_r. Suppose that there exists a positive integer R, and a positive constant ϵ such that*

$$\phi_r \le \frac{2k}{(k-2) - n_r^{-(\frac{1}{2}+\epsilon)}} \qquad \forall r \ge R,$$

where n_r is the order of G_r. Then, if $n_r \to \infty$ with r, we have $i(G_r) \to 0$ as $r \to \infty$.

Proof. Let $r \ge R$, so that

$$\phi_r \le \frac{2k}{(k-2) - n_r^{-(\frac{1}{2}+\epsilon)}}.$$

Then, by (3.1), if g_r is the genus of G_r,

$$\begin{aligned} g_r &= 1 + n_r \left(\frac{(k-2)\phi_r - 2k}{4\phi_r} \right) \\ &\le 1 + \frac{1}{4} n_r^{(\frac{1}{2}-\epsilon)}. \end{aligned}$$

Hence we have $g_r = o(\sqrt{n_r})$, so that, by Corollary 3.5, it follows that $i(G_r) \to 0$ as $r \to \infty$, as required. ∎

The following theorem is a consequence of the above result.

Theorem 3.7. *Let $\{G_r\}_{r=1}^{\infty}$ be a family of k-regular $(k \leq 6)$ linear magnifiers, with average face size sequence $\{\phi_r\}_{r=1}^{\infty}$. Then*

$$\limsup_{r\to\infty} \phi_r \geq \frac{2k}{k-2}.$$

Proof. From Lemma 3.6 we infer, since $i(G_r)$ does not tend to zero as r tends to infinity, that for any choice of positive integer R and a positive constant ϵ, there exists an integer $r \geq R$ such that

$$\phi_r \geq \frac{2k}{(k-2) - n_r^{-(\frac{1}{2}+\epsilon)}}$$

$$> \frac{2k}{k-2}$$

where n_r is the order of G_r. Hence there exist arbitrarily large r for which $\phi_r > 2k/(k-2)$, so that $\limsup_{r\to\infty} \phi_r \geq 2k/(k-2)$, as required. ∎

We now look at cubic graphs, and in particular those whose average face size is an integer.

Lemma 3.8. *Let G be a cubic graph of (necessarily) even order n with genus g and average face size ϕ. Suppose that ϕ is an integer greater than 6 and that*

$$n = 2p_1p_2\ldots p_r$$

where $2 \leq p_1 \leq p_2 \leq \cdots \leq p_r$ and the p_i are all prime. Then either $p_1 \leq 7$ or $g > n/p_1$.

Proof. Let G have f faces in any embedding on a surface of genus g. Then, by (3.1),

$$f = 2 - 2g + (p_1p_2\ldots p_r).$$

Suppose that $g \leq n/p_1$ and $p_1 \geq 11$. We shall show that this implies that $\phi \leq 6$ if ϕ is an integer. We have

$$f \geq 2 - 4(p_2\cdots p_r) + (p_1\cdots p_r),$$

and thus

$$\phi = \frac{3n}{f}$$

$$\leq \frac{6p_1 \dots p_r}{2 - 4(p_2 \dots p_r) + (p_1 \dots p_r)}.$$

From this we see that

$$\phi < \frac{6p_1}{p_1 - 4},$$

which is less than p_1 if $p_1 \geq 11$. Hence $p_1 > \phi$, so that $p_i > \phi$ for all i. Now the genus of G is given by Euler's formula to be

$$g = 1 + n\left(\frac{(k-2)\phi - 2k}{4\phi}\right)$$

$$= 1 + \frac{n}{4} - \frac{3n}{2\phi} \qquad \text{since } k = 3.$$

Thus, g being an integer, we have

$$\frac{3n}{\phi} = -2g + 2 + \frac{n}{2},$$

which means that $3n/\phi$ is an integer, so that $\phi | 6p_1 \dots p_r$. However, $\phi < p_i$ for all i, so it follows that $\phi | 6$. In particular, $\phi \leq 6$, whence the theorem follows. ∎

We can apply this result to families of graphs as follows.

Theorem 3.9. *Let* $\{G_r\}_{r=1}^{\infty}$ *be a family of cubic linear magnifiers. Let* G_r *have integral average face size* ϕ_r*, order* n_r *and genus* g_r*, for all* r*. Suppose*

$$n_r = 2p_1(r) \dots p_s(r),$$

where $2 \leq p_1(r) \leq \dots \leq p_s(r)$*, all* $p_i(r)$ *are prime, and* s *may depend on* r*. Then there* *exists a positive integer R such that*

$$\forall r \geq R \quad \textit{either} \quad (a) \quad p_1(r) \leq 7$$

$$\textit{or} \quad (b) \quad g_r > n_r/p_1(r).$$

Proof. Suppose that there exist arbitrarily large r for which $\phi_r \leq 6$. Then clearly for any positive constants α, ϵ there must be arbitrarily large r for which

$$\phi_r \leq \frac{6}{1 - \alpha n_r^{-(\frac{1}{2}+\epsilon)}}.$$

It follows, from Lemma 3.6, that there would be a subsequence $\{G_{m_r}\}_{r=1}^{\infty}$ of $\{G_r\}_{r=1}^{\infty}$ with the property that $i(G_{m_r}) \to 0$ as r tends to infinity. But this contradicts the fact that $\{G_r\}_{r=1}^{\infty}$ is a family of cubic magnifiers. Hence there must exist an integer R such that $\phi_r \geq 7$ for all $r \geq R$. The result then follows from Lemma 3.8. ∎

3.3 Finite Element Graphs

A *Finite Element Graph* (as defined in [**LT**], and henceforth abbreviated to FEG) is a graph G obtained from a plane embedding of a planar graph by adding all possible edges of the form $\{u, v\}$ where u and v are two non-adjacent vertices on the boundary of a common face in the embedded planar graph. In this section we investigate the magnifying properties of such graphs.

Lemma 3.10. *Let G be a FEG of order n such that no face of the underlying plane embedding has size greater than k. Then there exists a partition $V = A \cup B \cup C$ of the vertices of G with the following properties:*

(i) No edge of G joins a vertex in A to a vertex in B;

(ii) $|A|, |B| \leq 2n/3$;

(iii) $|C| \leq 4\lfloor k/2 \rfloor \sqrt{n}$.

Proof. Lipton and Tarjan [**LT**, Corollary 4]. ∎

Corollary 3.11. *Let $\{G_r\}_{r=1}^{\infty}$ be a family of k-regular FEGs, the order of G_r being n_r, and n_r tending to infinity with r. Then $i(G_r) \to 0$ as r tends to infinity.*

Proof. Let $A_r \cup B_r \cup C_r$ be the partition, as described in Lemma 3.10, of the vertices of G_r. Let $|A_r| \leq |B_r|$ so that $|A_r| \leq n_r/2$. Since G_r is k-regular, then every face of the underlying embedded planar graph has size at most $k+1$, from which we see that

$$|C_r| \leq 4\lfloor (k+1)/2 \rfloor \sqrt{n_r}.$$

Hence it is clear that

$$|\Gamma(A_r) \setminus A_r| \leq 4\lfloor (k+1)/2 \rfloor \sqrt{n_r}. \tag{3.2}$$

Now clearly $|A_r| + |B_r| \geq n_r - 4\lfloor (k+1)/2 \rfloor \sqrt{n_r}$, and this implies, together with the fact that $|B_r| \leq 2n_r/3$, that

$$|A_r| \geq \frac{n_r}{3} - 4\lfloor (k+1)/2 \rfloor \sqrt{n_r}, \tag{3.3}$$

which is positive for large enough r. Now $|A_r| \leq n_r/2$, with the result that $i(G_r) \leq i(A_r)$, so that (3.2) and (3.3) yield

$$i(G_r) \leq \frac{4\lfloor (k+1)/2 \rfloor \sqrt{n_r}}{(n_r/3) - 4\lfloor (k+1)/2 \rfloor \sqrt{n_r}}$$

$$= O(n_r^{-\frac{1}{2}}).$$

Hence $i(G_r) \to 0$ as $r \to \infty$, as required. ∎

The above result may be summarised by saying that FEGs cannot be used to make families of linear magnifiers.

Chapter Four

Hamiltonian and Shift Graphs

The following chapter is divided into two sections, the first of which looks at the magnifying and enlarging properties of cubic (that is, 3-regular) graphs which possess a Hamiltonian cycle. The second section presents a family of graphs which are generalisations of circulant graphs, and uses the techniques developed in the first section to derive their eigenvalues, information which will be useful in Chapter 5.

4.1 Hamiltonian Cubic Graphs

Throughout this section, G will denote a cubic graph which possesses a Hamiltonian cycle. It will be convenient to label the vertices of G with the elements of the ring $\mathbf{Z}_n$, where n is the order of G. Then we can define such a graph G by an involution σ in the symmetric group S_n (on the elements of $\mathbf{Z}_n$). To be precise, the graph $G(\sigma)$ is defined to have vertex set $\mathbf{Z}_n$ and edge set given by the following rule: the vertex x is adjacent to $x-1$, $x+1$ and $\sigma(x)$, for each $x \in \mathbf{Z}_n$ (all expressions reduced modulo n). Obviously we could generalise this to higher valency graphs by including more involutions, but for the purposes of this section we will concentrate on the cubic case.

The techniques used to study the graphs presented in this chapter will be primarily spectral, so that the emphasis will be on enlarging as opposed to magnifying properties. To begin, however, we present two results which relate the involution σ with the quantity $i(G(\sigma))$.

Theorem 4.1. *Let $G(\sigma)$ be a cubic Hamiltonian graph of (necessarily) even order n. We define $d(i, \sigma(i))$ to be the distance between the vertices i and $\sigma(i)$ in the cycle formed on $\{0, 1, \ldots, n-1\}$ by joining each vertex x to $x-1$ and $x+1$ (reduced modulo n). If we* *define the parameter $\hat{\sigma}$ of $G(\sigma)$ by*

$$\hat{\sigma} = \frac{1}{n}\sum_{i=0}^{n-1} d(i,\sigma(i)),$$

then

$$i(G(\sigma)) \leq \frac{2}{n}(2+\hat{\sigma}).$$

Proof. We consider subsets X_i of vertices of $G(\sigma)$ of size $n/2$, where

$$X_i = \{i, i+1, \ldots, i+\frac{n}{2}-1\} \qquad (0 \leq i \leq n-1)$$

and all elements are reduced modulo n.

We will calculate the average value

$$\sum_{i=0}^{n-1} |\Gamma(X_i) \setminus X_i|/n$$

and use this to bound $i(G(\sigma))$. Consider an arbitrary vertex j. There are three cases to consider, depending on the value of $d(j,\sigma(j))$. We will assume, without loss of generality, that $d(j,\sigma(j)) = j - \sigma(j)$.

(i) Suppose that $0 < d(j,\sigma(j)) < n/2$. Then $j \in \Gamma(X_i) \setminus X_i$ if and only if

a) $\sigma(j) \in X_i$ and $j \notin X_i$, or

b) $i = j+1$, or

c) $i + \frac{n}{2} - 1 = j - 1$.

Hence $j \in \Gamma(X_i) \setminus X_i$ if and only if

$$i \in \{\sigma(j) - \frac{n}{2} + 1, \sigma(j) - \frac{n}{2} + 2, \ldots, j - \frac{n}{2}\} \cup \{j+1\} \cup \{j + \frac{n}{2}\},$$

where all elements are reduced modulo n. But $j + \frac{n}{2} = j - \frac{n}{2}$ (modulo n), whilst $0 < j - \sigma(j) < n/2$ implies that $j + 1 \notin \{\sigma(j) - \frac{n}{2} + 1, \ldots, j - \frac{n}{2}\}$. Hence there are $1 + d(j,\sigma(j))$ values of i for which $j \in \Gamma(X_i) \setminus X_i$.

(ii) Suppose $j-\sigma(j)=n/2$. Then $j \in \Gamma(X_i)\setminus X_i$ if and only if $i \in \{j+1, j+2, \ldots, j+\frac{n}{2}\}$, so here there are precisely $d(j,\sigma(j))$ suitable values of i.

(iii) Suppose that $j=\sigma(j)$. Then $j \in \Gamma(X_i)\setminus X_i$ if and only if $i=j+1$ or $j+n/2$, so there are $2+d(j,\sigma(j))$ suitable values of i.

From the above we see that $j \in \Gamma(X_i)\setminus X_i$ for at most $2+d(j,\sigma(j))$ different i, and hence that

$$\frac{1}{n}\sum_{i=0}^{n-1}|\Gamma(X_i)\setminus X_i| \leq \frac{1}{n}\sum_{j=0}^{n-1}(2+d(j,\sigma(j))).$$

Hence we have

$$\begin{aligned}\frac{i(X_0)+\cdots+i(X_{n-1})}{n} &\leq \frac{2}{n^2}\sum_{j=0}^{n-1}(2+d(j,\sigma(j))) \\ &= \frac{2}{n}(2+\hat{\sigma}).\end{aligned}$$

Thus there must exist an i for which $i(X_i) \leq \frac{2}{n}(2+\hat{\sigma})$. Since $|X_i| = n/2$ implies that $i(G(\sigma)) \leq i(X_i)$, the required result follows at once. ∎

The above upper bound is easily seen to lead to an asymptotic result of the following nature.

Corollary 4.2. *Let $\{G_r(\sigma_r)\}_{r=1}^{\infty}$ be a family of cubic Hamiltonian graphs with n_r, the order of G_r, tending to infinity with r. Then $\hat{\sigma}_r = o(n_r)$ implies that $i(G_r(\sigma_r)) \to 0$ as $r \to \infty$.*

Proof. Immediate from Theorem 4.1. ∎

Corollary 4.2 implies that, for a family of cubic Hamiltonian graphs to be magnifiers (that is, for the isoperimetric number i not to tend to zero as the order of the graph tends to infinity), the average 'length' of the edges $\{i, \sigma_r(i)\}$ of $G_r(\sigma_r)$ must not tend to zero as $r \to \infty$. We shall see later in this section that this necessary condition is not sufficient. We now show, for cubic Hamiltonian graphs, how we may improve the upper bound for $i(G)$ from Theorem 2.1.

Theorem 4.3. *Suppose that $G(\sigma)$ is a simple, cubic Hamiltonian graph of order n, and that it has $\tau = \tau(G)$ triangles. Then*

$$i(G(\sigma)) \leq \frac{(n+4)}{4n(n-2)(n-4)}\left(n(3n-8) - \tau(n-4)\right).$$

Proof. We will write $\sigma = (\alpha_1\ \beta_1)\ldots(\alpha_{n/2}\ \beta_{n/2})$, and refer to an edge of $G(\sigma)$ of the form $\{\alpha_i, \beta_i\}$ as a *cross-edge* of $G(\sigma)$. We consider the set Φ of all subsets of vertices of size $2\lfloor n/4 \rfloor$ formed by the union of the end-points of $\lfloor n/4 \rfloor$ cross-edges, so that $|\Phi| = \binom{n/2}{\lfloor n/4 \rfloor}$ if $G(\sigma)$ is simple. We will evaluate the sum

$$S = \sum_{X \in \Phi} |\Gamma(X) \setminus X|.$$

A three-cycle in the graph $G(\sigma)$ will necessarily be of the form $(x-1)x(x+1)(x-1)$ for some $x \in \mathbf{Z}_n$, so that $\{x+1, x-1\}$ is a cross-edge. (Thus the number of triangles in $G(\sigma)$ is simply the number of transpositions $(\alpha_i\ \beta_i)$ in the above expression for σ which satisfy $d(\alpha_i, \beta_i) = 2$). Given $x \in VG(\sigma)$, how many $X \in \Phi$ satisfy $x \in \Gamma(X) \setminus X$? This occurs if and only if $x+1$ or $x-1$ is in X and $x \notin X$. Firstly, suppose that $\{x-1, x+1\}$ is *not* a cross-edge of $G(\sigma)$. Then

$$\begin{aligned}
\#\{\, X \in \Phi \mid x \in \Gamma(X) \setminus X \,\} &= \#\{\, X \in \Phi \mid x-1 \in X \quad \text{but} \quad x, x+1 \notin X \,\} \\
&\quad + \#\{\, X \in \Phi \mid x+1 \in X \quad \text{but} \quad x-1, x \notin X \,\} \\
&\quad + \#\{\, X \in \Phi \mid x-1, x+1 \in X \quad \text{but} \quad x \notin X \,\} \\
&= \binom{(n/2)-3}{\lfloor n/4 \rfloor - 1} + \binom{(n/2)-3}{\lfloor n/4 \rfloor - 1} + \binom{(n/2)-3}{\lfloor n/4 \rfloor - 2}. \qquad (4.1)
\end{aligned}$$

On the other hand, if $\{x-1, x+1\}$ is a cross-edge we see that

$$\#\{\, X \in \Phi \mid x \in \Gamma(X) \setminus X \,\} = \binom{(n/2)-2}{\lfloor n/4 \rfloor - 1}. \tag{4.2}$$

If we sum over all vertices x of $G(\sigma)$, we obtain, using (4.1) and (4.2),

$$S = 2(n-\tau)\binom{(n/2)-3}{\lfloor n/4\rfloor - 1} + (n-\tau)\binom{(n/2)-3}{\lfloor n/4\rfloor - 2} + \tau\binom{(n/2)-2}{\lfloor n/4\rfloor - 1}$$

$$= \frac{((n/2)-3)!}{(\lfloor n/4\rfloor - 1)!(\lceil n/4\rceil - 1)!}((n-\tau)(2\lceil n/4\rceil + \lfloor n/4\rfloor - 3) + \tau((n/2)-2)),$$

so that

$$i(G(\sigma)) \le \frac{S}{2\lfloor n/4\rfloor \binom{n/2}{\lfloor n/4\rfloor}}$$

$$= \frac{\lceil n/4\rceil}{n((n/2)-1)((n/2)-2)}\left(n\left(\frac{n}{2} + \left\lceil\frac{n}{4}\right\rceil - 3\right) - \tau\left(\left\lceil\frac{n}{4}\right\rceil - 1\right)\right).$$

Finally, using the inequalities $n/4 \le \lceil n/4\rceil \le (n/4)+1$ we deduce from the above that

$$i(G(\sigma)) \le \frac{(n+4)}{4n(n-2)(n-4)}(n(3n-8) - \tau(n-4)),$$

as required. ∎

Note that the above result supplies an asymptotic upper bound for $i(G(\sigma))$ (as the order tends to infinity) of $\frac{3}{4}$, compared to the value $\frac{7}{8}$ for general cubic graphs obtained from Corollary 2.2.

We now study a spectral approach to the class of cubic Hamiltonian graphs. If $G(\sigma)$ is such a graph we may write its adjacency matrix $A(\sigma)$ as

$$A(\sigma) = C_n + P(\sigma),$$

where $C_n = [c_{ij}]$ is an $n \times n$ matrix given by $c_{ij} = 1$ if $j = i \pm 1 \pmod{n}$, and $c_{ij} = 0$ otherwise. $P(\sigma)$ is the $n \times n$ permutation matrix of σ, so that

$$(P(\sigma))_{ij} = \begin{cases} 1 & (\sigma(i) = j); \\ 0 & (\text{otherwise}). \end{cases}$$

In what follows we will adapt the techniques of Jimbo and Maruoka [**JM**] to these graphs. We let $\Omega_n = [\omega_{ij}]$ denote the $n \times n$ matrix over the field $\mathbf{C}$ of complex numbers given by $\omega_{ij} = \omega^{ij}/\sqrt{n}$, where $\omega = \exp(2\pi \mathrm{i}/n)$.

Lemma 4.4. *If Ω_n^* denotes the complex conjugate transpose of Ω_n then $\Omega_n^* \Omega_n = I$.*

Proof.

$$\begin{aligned}(\Omega_n^* \Omega_n)_{ij} &= \sum_{k=0}^{n-1} \bar{\omega}_{ik} \omega_{kj} \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \omega^{(j-i)k} \\ &= \begin{cases} 1 & (j = i) \\ 0 & (j \neq i) \end{cases}\end{aligned}$$

as required. ∎

Lemma 4.5. *If $G(\sigma)$ is of order n, then*

$$\Omega_n^* A(\sigma) \Omega_n = \left(\begin{array}{c|c} 3 & \mathbf{0} \\ \hline \mathbf{0}^T & H(\sigma) \end{array} \right),$$

where $\mathbf{0}$ denotes the vector $\underbrace{(0, \ldots, 0)}_{n-1}$ and $\mathbf{0}^T$ is its transpose. Furthermore, $H(\sigma)$ is Hermitian of order $n-1$ and has largest eigenvalue equal to $\lambda_1(G(\sigma))$.

Proof. Since $A(\sigma)$ is a real symmetric matrix, then $A(\sigma)^* = A(\sigma)$, so that

$$(\Omega_n^* A(\sigma) \Omega_n)^* = \Omega_n^* A(\sigma) \Omega_n,$$

which shows that $\Omega_n^* A(\sigma)\Omega_n$ is Hermitian. Thus it must also be true that $H(\sigma)$ is Hermitian, provided that $\Omega_n^* A(\sigma)\Omega_n$ takes the form indicated.

Let $\Omega_n^* A(\sigma)\Omega_n = [\alpha_{ij}]$. Then clearly

$$\alpha_{00} = \sum_{k=0}^{n-1} \bar{\omega}_{k0}\left(\sum_{l=0}^{n-1} a_{kl}\omega_{l0}\right)$$

$$= \frac{1}{n}\sum_{k,l=0}^{n-1} a_{kl} \qquad \text{since } \omega_{k0} = n^{-1/2}$$

$$= 3$$

because $G(\sigma)$ is cubic.

If $j \neq 0$ we have

$$\alpha_{0j} = \sum_{k=0}^{n-1} \bar{\omega}_{k0}\left(\sum_{l=0}^{n-1} a_{kl}\omega_{lj}\right)$$

$$= \frac{1}{\sqrt{n}}\sum_{l=0}^{n-1} \omega_{lj}\left(\sum_{k=0}^{n-1} a_{kl}\right)$$

$$= \frac{3}{\sqrt{n}}\sum_{l=0}^{n-1} \omega_{lj}$$

$$= 0,$$

so that $\Omega_n^* A(\sigma)\Omega_n$ is of the required form. Since similar matrices have the same spectrum, the spectrum of $H(\sigma)$ is just that of $A(\sigma)$ with 3 removed once, and so the largest eigenvalue of $H(\sigma)$ is $\lambda_1(G(\sigma))$, by definition. ∎

Theorem 4.6. *If $G(\sigma)$ is a cubic Hamiltonian graph of order n, then $\lambda_1(G(\sigma))$ is equal to the largest eigenvalue of $H(\sigma) = [h_{ij}(\sigma)]$ $(1 \leq i, j \leq n-1)$, where*

$$h_{ij}(\sigma) = \begin{cases} 2\cos\frac{2\pi j}{n} + \frac{1}{n}\sum_{k=0}^{n-1}\omega^{j(\sigma(k)-k)} & \text{if } j = i; \\ \frac{1}{n}\sum_{k=0}^{n-1}\omega^{\sigma(k)j-ki} & \text{otherwise.} \end{cases}$$

Proof. We know that we may write the adjacency matrix of $G(\sigma)$ as

$$A(\sigma) = C_n + P(\sigma).$$

Thus

$$\Omega_n^* A(\sigma)\Omega_n = \Omega_n^* C_n \Omega_n + \Omega_n^* P(\sigma)\Omega_n,$$

and we determine $H(\sigma)$ by evaluating the two matrices on the right-hand side of the above equation. From the earlier definition of $C_n = [c_{kl}]$, we know that $c_{kl} = 1$ if and only if $k = l \pm 1$ and is 0 otherwise, so that

$$\left(\Omega_n^* C_n \Omega_n\right)_{ij} = \frac{1}{n}\sum_{k=0}^{n-1} \omega^{-ki}\left(\omega^{j(k+1)} + \omega^{j(k-1)}\right)$$

$$= \frac{1}{n}\left(\omega^j + \omega^{-j}\right)\sum_{k=0}^{n-1} \omega^{k(j-i)}.$$

Hence

$$\Omega_n^* C_n \Omega_n = \operatorname{diag}\left(2\cos\frac{2\pi j}{n}\right)_{j=0}^{n-1}, \tag{4.3}$$

where we define $\operatorname{diag}(\alpha_j)_{j=0}^{n-1}$ to be an $n \times n$ matrix $[\alpha_{ij}]$ with all off-diagonal entries equal to zero and $\alpha_{jj} = \alpha_j$ for $0 \leq j \leq n-1$.

From the earlier definition of $P(\sigma) = [p_{kl}(\sigma)]$, we know that $p_{kl}(\sigma) = 1$ if and only if $l = \sigma(k)$, and is zero otherwise. We thus have

$$\left(\Omega_n^* P(\sigma)\Omega_n\right)_{ij} = \frac{1}{n}\sum_{k=0}^{n-1} \omega^{\sigma(k)j - ki}. \tag{4.4}$$

Hence, by combining equations (4.3) and (4.4) together with the fact that

$$\Omega_n^* C_n \Omega_n + \Omega_n^* P(\sigma)\Omega_n = \left(\begin{array}{c|c} 3 & \mathbf{0} \\ \hline \mathbf{0}^T & H(\sigma) \end{array}\right),$$

it is easy to see that $H(\sigma)$ is of the required form. ∎

The result of Theorem 4.6 is useful in cases where $H(\sigma)$ is a sparse or 'nearly diagonal' matrix, where it may be possible to determine the spectrum of $G(\sigma)$ completely and so determine the enlarging behaviour as the order of the graph tends to infinity. We give two examples where this can be done.

(a) The class of graphs known as Möbius ladder graphs is defined by the involution (of the elements of $\mathbf{Z}_n$) $\sigma_n(k) = k + n/2$. Hence Theorem 4.6 gives us

$$\left(\Omega_n^* P(\sigma_n)\Omega_n\right)_{ij} = \frac{1}{n}\sum_{k=0}^{n-1} \omega^{(j-i)k}\omega^{\frac{nj}{2}}$$

$$= \begin{cases} (-1)^j & (j = i); \\ 0 & (j \neq i). \end{cases}$$

Hence

$$H(\sigma_n) = \operatorname{diag}\left(2\cos\frac{2\pi j}{n} + (-1)^j\right)_{j=1}^{n-1},$$

and the eigenvalues of the Möbius ladder graph $G(\sigma_n)$ are simply 3 together with the diagonal elements of $H(\sigma_n)$. In particular, it is clear that

$$3 \geq \lambda_1(G(\sigma_n)) = 2\cos\frac{4\pi}{n} + 1,$$

so that, as $n \to \infty$, the subdominant eigenvalue of $G(\sigma_n)$ tends to the value 3. We may summarise this by saying that there are no enlarging families of Möbius ladder graphs. Consequently, we have a counter-example to the converse statement of Corollary 4.2, for these graphs have the property that $\hat{\sigma}_n \neq o(n)$, but they do not supply families of linear enlargers (or, equivalently, magnifiers).

(b) Suppose that cubic Hamiltonian graph $G(\sigma_n)$ is defined by the involution (of the elements of $\mathbf{Z}_n$) $\sigma_n(k) = -k$. Then by using the result of Theorem 4.6 we have

$$(\Omega_n^* P(\sigma_n)\Omega_n)_{ij} = \sum_{k=0}^{n-1} \omega^{-k(j+i)}$$

$$= \begin{cases} 1 & (j \equiv -i \pmod n); \\ 0 & \text{(otherwise)}. \end{cases}$$

Thus

$$H(\sigma_n) = \operatorname{diag}\big(2\cos\frac{2\pi j}{n}\big)_{j=1}^{n-1} + \widehat{\operatorname{diag}}\big(1\big)_{j=1}^{n-1},$$

where $\widehat{\operatorname{diag}}\big(\alpha_j\big)_{j=1}^{n-1}$ denotes the $(n-1) \times (n-1)$ matrix $[\alpha_{ij}]$ for which $\alpha_{ij} = 0$ if $i \not\equiv -j \pmod n$, and $\alpha_{ij} = \alpha_j$ otherwise. In this case it is a simple matter to evaluate the eigenvalues of $H(\sigma_n)$, and they are of the form

$$\lambda_j = 2\cos\frac{2\pi j}{n} + 1 \qquad (1 \le j \le n-1).$$

Clearly this implies that the subdominant eigenvalue of $G(\sigma_n)$ satisfies

$$3 \ge \lambda_1(G(\sigma_n)) = 2\cos\frac{2\pi j}{n} + 1,$$

so that it must tend to 3 as n tends to infinity. Hence there are no enlarging families of these graphs.

Of considerably more interest than the preceding examples is the case where we take the order of the cubic Hamiltonian graph $G(\sigma_p)$ to be a prime p by defining its vertex set to be the Galois field $\mathbf{F}_p$, and define the involution σ_p by the rule

$$\sigma_p(k) = \begin{cases} 0 & (k=0); \\ k^{-1} & \text{otherwise}. \end{cases}$$

Using a numerical package on a computer to estimate $\lambda_1(G(\sigma_p))$ for various primes p seemed to offer some numerical evidence that the value would not approach 3 as $p \to \infty$, and hence hope that these graphs would supply a family of linear enlargers. Unfortunately no proof has been forthcoming, but it is of interest to note that Lubotzky, Phillips and Sarnak[**LPS2**] were able to show that a very closely related family of cubic graphs did form a family of linear enlargers. Their graphs X_p (p an odd prime) have as a vertex set the elements of the projective line $PG(1,p) = \{0, 1, \ldots, p-1, \infty\}$, with $\{x, y\}$ being an edge if and only if $x = -y^{-1}$ or $x = y \pm 1$. They were able to show that these graphs satisfied the inequality $\lambda_1(X_p) \le 2.9990$. Whilst this bound does not compare favourably with the bound for a cubic Ramanujan graph of $\lambda_1 \le 2\sqrt{2}$ (see Chapter 7 for the definition of a Ramanujan graph and a proof of this bound), these are certainly the simplest enlargers to have been constructed so far.

4.2 Shift Graphs

We begin this section by defining a class of graphs called *shift graphs* which includes a large number of Hamiltonian graphs amongst them. We will then proceed to calculate their spectra. We will not assume that our graphs are cubic, or indeed simple, but will assume that they all have *even* order.

Let n be a positive even integer. We let X_n, Y_n and Z_n be three multisets (that is, elements of X_n, Y_n and Z_n may occur with multiplicity greater than one) of integers modulo n. Further, we assume that they have the following form

$$X_n = \{\, x_{1n}, \ldots, x_{\alpha(n),n} \mid x_{jn} \equiv 0 \pmod 2 \quad \text{for} \quad 1 \le j \le \alpha(n) \,\}$$

$$Y_n = \{\, y_{1n}, \ldots, y_{\beta(n),n} \mid y_{jn} \equiv 1 \pmod 2 \quad \text{for} \quad 1 \le j \le \beta(n) \,\}$$

$$Z_n = \{\, z_{1n}, \ldots, z_{\gamma(n),n} \mid z_{jn} \equiv 1 \pmod 2 \quad \text{for} \quad 1 \le j \le \gamma(n) \,\},$$

where $\alpha(n), \beta(n), \gamma(n)$ are non-negative integers (if one of them is zero, then the corresponding multiset is defined to be empty) which satisfy

$$2\big(\alpha(n)+\beta(n)\big)+\gamma(n)=k. \tag{4.5}$$

If we denote the triplet (X_n, Y_n, Z_n) by T_n, then we define the *shift graph* $G_n(T_n)$ to be a graph of order n with vertex set $\mathbf{Z}_n = \{0, 1, \ldots, n-1\}$, such that a vertex $x \in \mathbf{Z}_n$ is adjacent to

(a) $x \pm x_{jn} \pmod n$ for $j = 1, \ldots, \alpha(n)$;

(b) $x \pm y_{jn} \pmod n$ for $j = 1, \ldots, \beta(n)$;

(c) $x + (-1)^x z_{jn} \pmod n$ for $j = 1, \ldots, \gamma(n)$.

From the above definition of adjacency and equation (4.5) it is obvious that the graph $G_n(T_n)$ is k-regular. We will make one further assumption, that the triplet T_n always satisfies $Y_n \cup Z_n \neq \emptyset$, for otherwise the graph would be disconnected. We will now derive the spectrum of $G_n(T_n)$, using the techniques introduced in the previous section.

For $1 \leq r \leq \alpha(n)$ the matrix C_r is defined by

$$(C_r)_{ij} = \begin{cases} 1 & (j \equiv i \pm x_{rn} \pmod n); \\ 0 & (\text{otherwise}). \end{cases}$$

The matrices D_r (for $1 \leq r \leq \beta(n)$) are defined analagously with respect to the parameters y_{rn}, and the matrices E_r (for $1 \leq r \leq \gamma(n)$) are given by

$$(E_r)_{ij} = \begin{cases} 1 & (j \equiv i + (-1)^i z_{rn} \pmod n); \\ 0 & (\text{otherwise}). \end{cases}$$

It follows that the adjacency matrix A_n of the graph $G_n(T_n)$ is given by

$$A_n = \sum_{r=1}^{\alpha(n)} C_r + \sum_{r=1}^{\beta(n)} D_r + \sum_{r=1}^{\gamma(n)} E_r. \tag{4.6}$$

If we define the matrix Ω_n in the same way as in the previous section, then we have the following result.

Lemma 4.7. *(i)* $\Omega_n^* C_\tau \Omega_n = \operatorname{diag}\left(2\cos\frac{2\pi x_{\tau n} j}{n}\right)_{j=0}^{n-1}$.

(ii) $\Omega_n^* D_\tau \Omega_n = \operatorname{diag}\left(2\cos\frac{2\pi y_{\tau n} j}{n}\right)_{j=0}^{n-1}$.

(iii)

$$\left(\Omega_n^* E_\tau \Omega_n\right)_{ij} = \begin{cases} \cos\frac{2\pi z_{\tau n} j}{n} & (i = j); \\ \mathrm{i}\sin\frac{2\pi z_{\tau n} j}{n} & (i - j \equiv n/2 \pmod{n}); \\ 0 & (\textit{otherwise}). \end{cases}$$

Proof. (i) From the definition of the matrices involved we have, if $C_\tau = [c_{ij}^\tau]$,

$$\left(\Omega_n^* C_\tau \Omega_n\right)_{ij} = \frac{1}{n}\sum_{k=0}^{n-1} \bar{\omega}_{ki}\left(\sum_{l=0}^{n-1} c_{kl}^\tau w_{lj}\right)$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} \omega^{-ki}\left(\omega^{j(k+x_{\tau n})} + \omega^{j(k-x_{\tau n})}\right)$$

$$= \frac{1}{n}\left(\omega^{jx_{\tau n}} + \omega^{-jx_{\tau n}}\right)\sum_{k=0}^{n-1} \omega^{k(j-i)}$$

$$= \begin{cases} 2\cos\frac{2\pi j x_{\tau n}}{n} & (j = i); \\ 0 & (\text{otherwise}). \end{cases}$$

The proof of (ii) is very similar.

(iii) Let $E_\tau = [e_{ij}^\tau]$, and $\sigma_{\tau n}$ denote the involution of $\mathbf{Z}_n$ given by $\sigma_{\tau n} \equiv x + (-1)^x z_{\tau n}$ (mod n). Then

$$\left(\Omega_n^* E_\tau \Omega_n\right)_{ij} = \frac{1}{n}\sum_{k=0}^{n-1} \bar{\omega}_{ki}\left(\sum_{l=0}^{n-1} e_{kl}^\tau \omega_{lj}\right)$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} \omega^{-ki}\omega^{\sigma_{\tau n}(k)j}.$$

Thus

$$\left(\Omega_n^* E_r \Omega_n\right)_{ij} = \frac{1}{n} \sum_{r=0}^{(n/2)-1} \omega^{(2r+z_{rn})j-2ri} + \frac{1}{n} \sum_{r=0}^{(n/2)-1} \omega^{(2r+1-z_{rn})j-(2r+1)i}$$

$$= \frac{1}{n}\left(\omega^{z_{rn}j} + \omega^{(j-i)}\omega^{-z_{rn}j}\right) \sum_{r=0}^{(n/2)-1} \omega^{2r(j-i)}$$

$$= \begin{cases} \cos\frac{2\pi z_{rn}j}{n} & (i=j); \\ \mathbf{i}\sin\frac{2\pi z_{rn}j}{n} & (i-j \equiv n/2 \pmod n); \\ 0 & (\text{otherwise}). \end{cases}$$

This completes the proof. ∎

Corollary 4.8. *(i) If A_n is the adjacency matrix of $G_n(T_n)$, then*

$$\Omega_n^* A_n \Omega_n = \left(\begin{array}{c|c} k & \mathbf{0} \\ \hline \mathbf{0}^T & H_{n-1} \end{array}\right),$$

where H_{n-1} is Hermitian with largest eigenvalue $\lambda_1(G_n(T_n))$.

(ii) The matrix $H_{n-1} = [h_{ij}]$ (for $1 \le i, j \le n-1$) is given by

$$h_{ij} = \begin{cases} 2\sum_{r=1}^{\alpha(n)} \cos\frac{2\pi x_{rn}j}{n} + 2\sum_{r=1}^{\beta(n)} \cos\frac{2\pi y_{rn}j}{n} & \\ \qquad + \sum_{r=1}^{\gamma(n)} \cos\frac{2\pi z_{rn}j}{n} & (i=j); \\ \mathbf{i}\sum_{r=1}^{\gamma(n)} \sin\frac{2\pi z_{rn}j}{n} & (i-j \equiv n/2 \pmod n); \\ 0 & (\text{otherwise}). \end{cases}$$

Proof. (i) The proof is analagous to that of Lemma 4.5.

(ii) This follows directly from (i), the expression for the adjacency matrix given by equation (4.6), and the results of Lemma 4.7. ∎

In order to present the matrix H_{n-1} in a more manageable form, we define the functions f, g and h by

$$f(r) = 2\sum_{\tau=1}^{\alpha(n)} \cos\frac{2\pi x_{\tau n} r}{n} + 2\sum_{\tau=1}^{\beta(n)} \cos\frac{2\pi y_{\tau n} r}{n} + \sum_{\tau=1}^{\gamma(n)} \cos\frac{2\pi z_{\tau n} r}{n}$$

$$\begin{aligned} g(r) &= f((n/2)+r) \\ &= 2\sum_{\tau=1}^{\alpha(n)} \cos\frac{2\pi x_{\tau n} r}{n} - 2\sum_{\tau=1}^{\beta(n)} \cos\frac{2\pi y_{\tau n} r}{n} - \sum_{\tau=1}^{\gamma(n)} \cos\frac{2\pi z_{\tau n} r}{n} \end{aligned}$$

$$h(r) = \mathrm{i}\sum_{\tau=1}^{\gamma(n)} \sin\frac{2\pi z_{\tau n} r}{n}.$$

Note that, since each $z_{\tau n}$ is odd, we have $h((n/2)+r) = -h(r)$. Then, by using the result of Corollary 4.8, we see that H_{n-1} has the form shown below.

$$H_{n-1} = \begin{pmatrix} f(1) & 0 & \cdots & 0 & 0 & -h(1) & 0 & \cdots & 0 \\ 0 & f(2) & \cdots & 0 & 0 & 0 & -h(2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f(\frac{n}{2}-1) & 0 & 0 & 0 & \cdots & -h(\frac{n}{2}-1) \\ 0 & 0 & \cdots & 0 & f(\frac{n}{2}) & 0 & 0 & \cdots & 0 \\ h(1) & 0 & \cdots & 0 & 0 & g(1) & 0 & \cdots & 0 \\ 0 & h(2) & \cdots & 0 & 0 & 0 & g(2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h(\frac{n}{2}-1) & 0 & 0 & 0 & \cdots & g(\frac{n}{2}-1) \end{pmatrix}.$$

Then it is not too hard to see that, by suitable row and column operations,

$$\det(\lambda I - H_{n-1}) = \big(\lambda - f(n/2)\big) \prod_{r=1}^{(n/2)-1} \Big(\big(\lambda - f(r)\big)\big(\lambda - g(r)\big) + h(r)^2 \Big).$$

Now

$$\big(\lambda - f(r)\big)\big(\lambda - g(r)\big) + h(r)^2 = \lambda^2 - \big(f(r)+g(r)\big)\lambda + \big(f(r)g(r) + h(r)^2\big),$$

which has roots

$$\lambda_r^+, \lambda_r^- = \frac{\big(f(r)+g(r)\big) \pm \sqrt{\big(f(r)-g(r)\big)^2 - 4h(r)^2}}{2}.$$

We have now determined the spectrum of $G_n(T_n)$ completely, and the results are summarised in the following theorem.

Theorem 4.9. *The spectrum of $G_n(T_n)$ is $\{\lambda_0, \lambda_{\frac{n}{2}}\} \cup \{\lambda_r^-, \lambda_r^+ \mid r = 1, 2, \ldots, \frac{n}{2} - 1\}$, where*

$$\lambda_0 = k$$

$$\begin{aligned} \lambda_{\frac{n}{2}} &= f(n/2) \\ &= 2\big(\alpha(n) - \beta(n)\big) - \gamma(n) \end{aligned}$$

$$\begin{aligned} \lambda_r^-, \lambda_r^+ = {} & 2 \sum_{\tau=1}^{\alpha(n)} \cos \frac{2\pi x_{\tau n} r}{n} \\ & \pm \sqrt{\left(2 \sum_{\tau=1}^{\beta(n)} \cos \frac{2\pi y_{\tau n} r}{n} + \sum_{\tau=1}^{\gamma(n)} \cos \frac{2\pi z_{\tau n} r}{n}\right)^2 + \left(\sum_{\tau=1}^{\gamma(n)} \sin \frac{2\pi z_{\tau n} r}{n}\right)^2}, \end{aligned}$$

for $r = 1, 2, \ldots, (n/2) - 1$. ■

We will use this result in the next chapter to examine the enlarging properties of shift graphs.

Chapter Five

Some Consequences of a Theorem of Klawe

5.1 Klawe Graphs and Circulant Graphs

In what follows we state the result of Klawe on non-expanding families of graphs (that is, those families whose graphs have an expansion that tends to zero as their order tends to infinity). We will use this result to derive a 'technical' lemma, which will be seen to apply to different classes of graphs than those covered by Klawe's Theorem. These will include the Shift graphs introduced in Chapter 4, as well as some classes of Cayley graphs, which will be discussed in Chapter 6. As in the last chapter, we do not assume the simplicity of any graph. Let

$$\Psi_n = \{ f_i : x \mapsto a_i x + b_i \mid a_i, b_i \in \mathbf{Z}_n \quad 1 \leq i \leq k \}$$

denote a multiset of k one-dimensional affine transformationsof the ring $\mathbf{Z}_n$. The bipartite graph $G_n(\Psi_n)$ is defined on a set of input vertices $I_n = \{ x(i) \mid 1 \leq i \leq n \}$ and output vertices $O_n = \{ y(i) \mid 1 \leq i \leq n \}$, in which there are r edges between $x(i)$ and $y(j)$, where r is the number of $f \in \Psi_n$ such that $j \equiv f(i) \pmod{n}$. We will refer in future to graphs of the form $G_n(\Psi_n)$ as *Klawe graphs.* The following, which we shall call Klawe's Theorem, shows that families of k-regular Klawe graphs never form families of linear expanders.

Theorem 5.1. *Suppose that Ψ_n is given for infinitely many n and has fixed cardinality k, independent of n. Then there exist functions $N(k)$ and $\delta(k,n)$ such that:*

(i) For every $n \geq N(k)$ there exists an $X \subseteq I_n$, in the graph $G_n(\Psi_n)$, such that $|X| \leq n/2$ and

$$|\Gamma(X)| < \left(1 + 2\delta(k,n)\left(1 - \frac{|X|}{n}\right)\right)|X|;$$

(ii) $\delta(k,n) \to 0$ as $n \to \infty$.

Proof. Klawe [**Kl**, Theorem 2.1]. ∎

Theorem 5.1(i) shows that the expansion d_n of a Klawe graph $G_n(\Psi_n)$ satisfies $d_n \leq 2\delta(k,n)$, and hence part (ii) shows that $d_n \to 0$ as $n \to \infty$. Hence no families of Klawe graphs are families of linear expanders.

We now go on to look at a class of (not necessarily bipartite) graphs and show that their *augmented double covers* (see Chapter 1 for a definition) are Klawe graphs.

Definition 5.2. *A (not necessarily simple) graph $G = (V,E)$ with $V = \{1,2,\ldots,n\}$ is said to be* circulant *if its adjacency matrix $A = [a_{ij}]$ satisfies $a_{ij} = a_{1,j-i+1}$, where the subscripts are reduced modulo n to lie in V.* ∎

Hence for such a graph G, the adjacency matrix A is completely determined by its first row $(a_1\, a_2\, \ldots\, a_n)$ and, since A is symmetric (because G is undirected) we have $a_i = a_{n-i+2}$ for $2 \leq i \leq n$. It is a simple matter to determine the spectrum of G.

Lemma 5.3. *If G is a circulant graph whose adjacency matrix is A, the first row of which is $(a_1\, a_2\, \ldots\, a_n)$, then*

$$\operatorname{spec}(G) = \{ \sum_{j=1}^{n} a_j \omega^{(j-1)r} \mid r = 0,1,\ldots,n-1 \}$$

where $\omega = \exp(2\pi \mathrm{i}/n)$.

Proof. Biggs [**Bi**, Proposition 3.5]. ∎

Given a circulant graph G of order n, it is clear that we may specify it completely by a multiset $S(G)$ of integers in $\{0,1,\ldots,n-1\}$, where $s \in S(G)$ with multiplicity a_{s+1}, for $s = 0,1,\ldots,n-1$. Then, mindful of the fact that all eigenvalues of G are real because A is symmetric, this leads us to the following expression for the spectrum of G using the result of Lemma 5.3.

Lemma 5.4. *If G is circulant of order n then*

$$\operatorname{spec}(G) = \{ \sum_{s \in S(G)} \cos \frac{2\pi sr}{n} \mid r = 0, 1, \ldots, n-1 \}. \blacksquare$$

The importance of circulant graphs lies in the fact that they can be used to 'model' certain types of spectra, as we now show.

Theorem 5.5. *Let F_n be a multiset of integers from the set $\{ 0, 1, \ldots, \lfloor n/2 \rfloor \}$, with m_i being the multiplicity of i in F_n. Then there exists a circulant graph G of order n and valency $2|F_n|$ such that*

$$\operatorname{spec}(G) = \{ 2 \sum_{\alpha \in F_n} \cos \frac{2\pi \alpha r}{n} \mid r = 0, 1, \ldots, n-1 \}.$$

Proof. Define the circulant matrix A by its first row as

$$a_1 = 2m_0$$

$$a_i = a_{n-i+2} = m_{i-1} \qquad (2 \le i \le \lceil n/2 \rceil)$$

$$a_{(n/2)+1} = 2m_{(n/2)},$$

the last definition being made if and only if n is even. Then A is the adjacency matrix of a circulant graph G of order n and valency k, where

$$k = a_1 + a_2 + \cdots + a_n$$

$$= 2 \sum_{i=0}^{\lfloor n/2 \rfloor} m_i$$

$$= 2|F_n|.$$

Then $\mathrm{spec}(G) = \{\lambda_0, \dots, \lambda_{n-1}\}$, where, by Lemma 5.4,

$$\lambda_r = \sum_{s \in S(G)} \cos \frac{2\pi sr}{n}$$

$$= 2m_0 \cos 0 + \sum_{i=2}^{\lceil n/2 \rceil} m_{i-1} \cos \frac{2\pi(i-1)r}{n} + 2m_{n/2} \cos \pi r$$

$$+ \sum_{i=2}^{\lceil n/2 \rceil} m_{i-1} \cos \frac{2\pi(n-i+1)r}{n}$$

$$= 2m_0 \cos 0 + 2 \sum_{i=1}^{\lfloor n/2 \rfloor} m_i \cos \frac{2\pi ir}{n}$$

$$= 2 \sum_{\alpha \in F_n} \cos \frac{2\pi \alpha r}{n}$$

as required. ∎

Now we consider the augmented double cover $\tilde{G}$ of a circulant graph G of order n. The graph $\tilde{G}$ is bipartite on input set $I_n = \{x(i) \mid 1 \le i \le n\}$ and output set $O_n = \{y(j) \mid 1 \le j \le n\}$. Suppose that there are r edges between i and j in G. Then, if $i \neq j$, there are r edges between $x(i)$ and $y(j)$ in $\tilde{G}$, whilst if $i = j$ there are $r+1$ edges between $x(i)$ and $y(j)$ in $\tilde{G}$. But $\{i, j\}$ is an edge of G if and only if there exists an $s \in S(G)$ such that $j \equiv i + s \pmod{n}$. From this it is clear that $\tilde{G} = G_n(\Psi_n)$ where

$$\Psi_n = \{ f_{id}: x \mapsto x \} \cup \{ f_s: x \mapsto x + s \mid s \in S(G) \}.$$

Or, put more simply:

Theorem 5.6. *If G is a circulant graph of order n and valency k, then the augmented double cover $\tilde{G}$ is a Klawe graph of order $2n$ and valency $k+1$.* ∎

Corollary 5.7. *Let $\{\tilde{G}_r\}_{r=1}^{\infty}$ be a family of augmented double covers of k-regular circulant graphs, with order n_r tending to infinity with r. Then, if $\tilde{G}_r$ is an $(n_r, k+1, d_r)$-expander, we have $d_r \to 0$ as $r \to \infty$.*

Proof. This is immediate from Theorems 5.1 and 5.6. ∎

From this we obtain the result which will be of considerable use throughout the rest of this chapter and the next.

Theorem 5.8. *Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of multisets such that each member of F_n lies in the set $\{0, 1, \ldots, n-1\}$, and $|F_n| \leq k$ for all n, where k is a constant. Let ϵ be a chosen positive constant. Then there is a positive integer $N = N(\epsilon)$ with the following property: if $n \geq N$ then there exists an integer $r_n \in \{1, 2, \ldots, n-1\}$ such that*

$$\cos \frac{2\pi\alpha r_n}{n} \geq 1 - \epsilon \qquad \forall \alpha \in F_n.$$

Proof. Since the result will hold for $\{F_n\}_{n=1}^{\infty}$ if and only if it holds for the sequence $\{F'_n\}_{n=1}^{\infty}$, where

$$F'_n = F_n \cup \underbrace{\{0, \ldots, 0\}}_{k-|F_n|},$$

then without loss of generality we may assume that $|F_n| = k$ for all n. We define the parameter

$$\phi(F_n) = \max_{1 \leq r \leq n-1} 2 \sum_{\alpha \in F_n} \cos \frac{2\pi\alpha r}{n}.$$

Since $\cos(2\pi(n-\alpha)r/n) = \cos(2\pi\alpha r/n)$ for integer r, we may assume that each $\alpha \in F_n$ lies in the set $\{0, 1, \ldots, \lfloor n/2 \rfloor\}$. Hence, by Theorem 5.5, there exists a circulant graph G_n of order n and valency $2k$ such that

$$\text{spec}(G_n) = \{\, 2 \sum_{\alpha \in F_n} \cos \frac{2\pi\alpha r}{n} \mid r = 0, 1, \ldots, n-1 \,\}.$$

Now combining the results of Corollary 5.7, Lemma 1.5 and Lemma 1.3(i), we deduce that $\lambda_1(G_n) \to 2k$ as $n \to \infty$. Now the value $r = 0$ corresponds to the eigenvalue $2k$, so that

$$\lambda_1(G_n) = \phi(F_n).$$

$N = N(\epsilon)$ such that, for any even $n \geq N$, there is an integer $r_n \in \{1, 2, \ldots, n-1\}$ with the property that

$$\cos \frac{2\pi\alpha r_n}{n} \geq 1 - \epsilon \qquad \forall \alpha \in F_n.$$

Now α is an integer, so that $\cos(2\pi\alpha r_n/n) = \cos(2\pi\alpha(n - r_n)/n)$ for all $\alpha \in F_n$. In other words, we may choose the integer r_n from the set $\{1, 2, \ldots, (n/2) - 1\}$. (The choice $r_n = n/2$ is ruled out because it would imply that $\cos \pi\alpha \geq 1 - \epsilon$ for all $\alpha \in F_n$. But we included in our definition of a shift graph the condition that $Y_n \cup Z_n \neq \emptyset$, and $\alpha \in Y_n \cup Z_n$ implies that $\cos \pi\alpha = -1$, which gives us a contradiction if we choose $\epsilon < 2$.)

Hence, using the expression for the eigenvalues of $G_n(T_n)$ given above, we deduce that $G_n(T_n)$ has an eigenvalue $\lambda^+_{r_n}$ such that

$$k \geq \lambda^+_{r_n} \geq 2(1-\epsilon)\alpha(n) + \sqrt{(1-\epsilon)^2\big(2\beta(n) + \gamma(n)\big)^2}$$

$$= (1-\epsilon)\big(2(\alpha(n) + \beta(n)) + \gamma(n)\big)$$

$$= (1-\epsilon)k,$$

so that $\lambda^+_{r_n} \to k$ as n tends to infinity. Consequently, because

$$\lambda_1(G_n(T_n)) \geq \max_{1 \leq r \leq (n/2)-1} \lambda^+_r,$$

then we deduce that $\lambda_1(G_n(T_n)) \to k$ as $n \to \infty$, as required. ∎

In particular, if we choose the triplet T_n so that $1 \in Y_n$ then the graph $G_n(T_n)$ is obviously Hamiltonian, so Theorem 5.9 supplies us with a large number of (not necessarily cubic) linear families of Hamiltonian graphs which do not enlarge (that is, the subdominant eigenvalue of these graphs tends to the valency k as their order tends to infinity).

In the next chapter we will use Theorem 5.8 to show the non-existence of enlarging linear families of certain classes of Cayley graphs.

Chapter Six

Non-Enlarging Families of Cayley Graphs

6.1 Cayley Graphs

Let G be a finite group, Ω a multiset of elements of G which generate G such that $\Omega = \Omega^{-1}$. The *Cayley Graph* $G(\Omega)$ has vertex set G, with there being r edges between vertices g and h if and only if r is the multiplicity of the element $g^{-1}h$ in Ω. The graph $G(\Omega)$ is undirected because $\Omega = \Omega^{-1}$ implies that $g^{-1}h \in \Omega$ (with multiplicity r) if and only if $h^{-1}g \in \Omega$ (with multiplicity r). For the purposes of this chapter we will not assume that Ω is a set (so that $G(\Omega)$ may have multiple edges), nor that $1 \notin \Omega$ (if $1 \in \Omega$ then the graph $G(\Omega)$ will have loops), but we always assume Ω generates G, so that the graph $G(\Omega)$ is connected. If $|\Omega| = k$ then $G(\Omega)$ will be k-regular.

Cayley graphs are of some interest in the field of enlarging or expanding families of graphs. The construction of optimal enlargers by Lubotzky, Phillips and Sarnak [**LPS**] uses Cayley graphs of $PGL(2, \mathbf{Z}_q)$. However in this chapter we will use the results of Chapter 5 together with some character theory to show that certain families of groups will never yield enlarging families of Cayley graphs. A suitable reference for the character theory involved is Ledermann [**Le**].

6.2 Abelian Groups

In this section we will prove that there are no enlarging linear families of Cayley graphs of Abelian groups. This result has already been proved, for example by Alon and Milman, but we include this proof because it uses the result derived from Klawe's Theorem in Chapter 5. Throughout $G(\Omega)$ will denote the Cayley graph of the group G with respect to the generating multiset Ω. First of all, we recall the structure theorem for finite Abelian groups.

Theorem 6.1. *Let G be a finite Abelian group. Then G is isomorphic to a direct sum*

$$G \cong \mathbf{Z}_{r_1} \oplus \cdots \oplus \mathbf{Z}_{r_\alpha}$$

of finite non-trivial cyclic groups of order r_i for $i = 1, 2, \ldots, \alpha$, such that $r_1 | r_2 | \ldots | r_\alpha$. The r_i are uniquely determined by G, and are called the torsion invariants *of G.* ∎

Next, some information on the irreducible characters of a finite Abelian group.

Theorem 6.2. *Let $G = \mathbf{Z}_{r_1} \oplus \cdots \oplus \mathbf{Z}_{r_\alpha}$ be an arbitrary finite Abelian group. For any element $\mathbf{g} = (g_1, g_2, \ldots, g_\alpha) \in G$ the function $\chi^{[\mathbf{g}]}: G \to \mathbf{C}$ defined by*

$$\chi^{[\mathbf{g}]}(\mathbf{x}) = \exp\left(2\pi\mathrm{i}\sum_{i=1}^{\alpha} \frac{g_i x_i}{r_i}\right),$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_\alpha)$, is an irreducible character of G of degree one. Furthermore the set $\{ \chi^{[\mathbf{g}]} \mid \mathbf{g} \in G \}$ is precisely the set of all $|G|$ distinct irreducible characters of G.

Proof. Ledermann [**Le**, section 2.4]. ∎

As a final preliminary, there is a result relating the irreducible characters of an *arbitrary* finite group G to the spectrum of the Cayley graph $G(\Omega)$.

Theorem 6.3. *Let G be a finite group of order n whose irreducible (complex) characters are $\chi_1, \ldots, \chi_h$ with respective degrees $n_1, \ldots, n_h$ ($\sum_{i=1}^{h} n_i^2 = n$). Then the spectrum of the Cayley graph $G(\Omega)$ may be written*

$$\mathrm{spec}(G(\Omega)) = \{ \lambda_{ijk} \mid 1 \le j, k \le n_i \quad 1 \le i \le h \},$$

where $\lambda_{ij1} = \cdots = \lambda_{ijn_i}$ (this common value being denoted by λ_{ij}) and, for any natural number t,

$$\lambda_{i1}^t + \cdots + \lambda_{in_i}^t = \sum_{g_1, \ldots, g_t \in \Omega} \chi_i\left(\prod_{s=1}^{t} g_s\right).$$

The above sum is taken over all t-tuples $(g_1, \ldots, g_t)$ of elements of Ω, where an element g may appear in as many co-ordinate places as its multiplicity in Ω.

Proof. Babai [**B**]. ∎

To prove the main result of this section, the following lemma is required.

Lemma 6.4. *Let $G(\Omega)$ be a k-regular Cayley graph (assumed, as always, to be connected), where G is an Abelian group whose decomposition (given by Theorem 6.1) is*

$$G = \mathbf{Z}_{r_1} \oplus \cdots \oplus \mathbf{Z}_{r_\alpha}.$$

Then any positive integer r can appear at most k times in the sequence $r_1, \ldots, r_\alpha$.

Proof. Suppose, for a contradiction, that the integer r appears at least $k+1$ times in the sequence $r_1, \ldots, r_\alpha$. Then, without any loss of generality, we may suppose that $r = r_1 = \cdots = r_{k+1}$. It follows that $\mathbf{Z}_r^{k+1} \leq G$. If we restrict the elements of Ω to the first $k+1$ co-ordinate positions, then (since Ω generates G) the resulting multiset $\hat{\Omega}$ must generate $\mathbf{Z}_r^{k+1}$, and has size k. Hence, since each element of $\hat{\Omega}$ has order at most r, we have

$$r^{k+1} = |\mathbf{Z}_r^{k+1}| = |\langle\hat{\Omega}\rangle| \leq r^k,$$

which is an obvious contradiction. This completes the proof. ∎

Theorem 6.5. *Let $\{G_n(\Omega_n)\}_{n=1}^{\infty}$ be a family of k-regular Cayley graphs of Abelian groups, whose orders tend to infinity with n. Then $\lambda_1(G_n(\Omega_n)) \to k$ as $n \to \infty$.*

Proof. Without loss of generality, we may assume from Theorem 6.1 that for each n we have

$$G_n = \mathbf{Z}_{r_1(n)} \oplus \cdots \oplus \mathbf{Z}_{r_{\alpha(n)}(n)},$$

where $r_1(n)|r_2(n)|\cdots|r_{\alpha(n)}(n)$ and we set $r(n) = r_{\alpha(n)}(n)$. Now $|\Omega_n| = k$ for all n, but $|G_n| \to \infty$ with n. Since, by Lemma 6.4, no integer can appear in the sequence of torsion invariants for G_n more than k times, then it must be true that $r(n) \to \infty$ as $n \to \infty$.

Using Theorems 6.2 and 6.3, we know that for each $\mathrm{g} \in G_n$ there is an eigenvalue of $G_n(\Omega_n)$ given by

$$\lambda_{\mathrm{g}}^{(n)} = \sum_{\omega \in \Omega_n} \chi^{[\mathrm{g}]}(\omega),$$

and these are the only eigenvalues of $G_n(\Omega_n)$. The case $\mathrm{g} = (0, \ldots, 0)$ clearly gives the eigenvalue k, so that

$$\lambda_1(G_n(\Omega_n)) = \max_{\mathrm{g} \in G_n \setminus \{0\}} \sum_{\omega \in \Omega_n} \exp\left(2\pi \mathrm{i} \sum_{i=1}^{\alpha(n)} \frac{\omega_i g_i}{r_i(n)}\right),$$

where $\omega = (\omega_1, \ldots, \omega_{\alpha(n)})$ and $\mathrm{g} = (g_1, \ldots, g_{\alpha(n)})$. In particular, if $H_n \subseteq G_n \setminus \{\mathbf{0}\}$ is given by

$$H_n = \{\, (0, 0, \ldots, 0, s) \mid 1 \leq s \leq r(n) - 1 \,\},$$

then

$$\lambda_1(G_n(\Omega_n)) \geq \max_{\mathrm{g} \in H_n} \sum_{\omega \in \Omega_n} \exp\left(2\pi \mathrm{i} \sum_{i=1}^{\alpha(n)} \frac{\omega_i g_i}{r_i(n)}\right)$$

$$= \max_{1 \leq s \leq r(n)-1} \sum_{\omega \in \Omega_n} \exp\left(\frac{2\pi \mathrm{i} \omega_{\alpha(n)} s}{r(n)}\right).$$

Each sum on the right-hand side of the above equation is an eigenvalue of the undirected graph $G_n(\Omega_n)$ and therefore real, so that

$$\lambda_1(G_n(\Omega_n)) \geq \max_{1 \leq s \leq r(n)-1} \sum_{\omega \in \Omega_n} \cos \frac{2\pi \omega_{\alpha(n)} s}{r(n)}.$$

We can now apply Theorem 5.8. For each n we define a multiset of integers from $\mathbf{Z}_{r(n)}$ by

$$F_{r(n)} = \{\, \omega_{\alpha(n)} \mid \omega \in \Omega_n \,\}$$

so that $|F_{r(n)}| = |\Omega_n| = k$ for all n. Since $r(n) \to \infty$ with n, then by Theorem 5.8,

$$\max_{1 \leq s \leq r(n)-1} \sum_{\beta \in F_{r(n)}} \cos\left(\frac{2\pi \beta s}{r(n)}\right) \to k$$

as $n \to \infty$. This implies that $\lambda_1(G_n(\Omega_n)) \to k$ as $n \to \infty$, as required. ∎

6.3 Dihedral Groups

In this section we prove that Cayley graphs of Dihedral groups do not form families of linear enlargers. For the purposes of simplicity, we consider the Dihedral groups

$$D_n = \langle\, \alpha, \beta \mid \alpha^2 = \beta^n = (\alpha\beta)^2 = 1 \,\rangle$$

for *odd* n only, since the proof for even n is very similar.

Consider the Dihedral group D_{2m+1}. An element of the form β^r for some integer r will be called a *rotation*. Otherwise it will necessarily be of the form $\alpha\beta^r$ for some integer r, and will be called a *reflection*.

Lemma 6.6.

(i) $(\alpha\beta^r)^{-1} = \alpha\beta^r$;

(ii) $\beta^r\alpha\beta^s = \alpha\beta^{s-r}$;

(iii) $\alpha\beta^r\alpha\beta^s = \beta^{s-r}$.

Proof. All immediate from the group axioms and defining relations for D_{2m+1}. ∎

We consider the Cayley graph $D_{2m+1}(\Omega_{2m+1})$, where Ω_{2m+1} is a multiset of generators of D_{2m+1} of size k satisfying $\Omega_{2m+1}^{-1} = \Omega_{2m+1}$. Suppose there are s_m rotations in Ω_{2m+1}. They fall into inverse pairs $\{\beta^r, \beta^{-r}\}$. If the multiplicity of β^r in Ω_{2m+1} is τ_r, then

$$s_m = \sum_{r=0}^{2m} \tau_r. \tag{6.1}$$

There are then $k - s_m$ reflections in Ω_{2m+1}, and each is self-inverse by Lemma 6.6(i), so that if the multiplicity of $\alpha\beta^r$ in Ω_{2m+1} is κ_r then

$$k - s_m = \sum_{r=0}^{2m} \kappa_r. \tag{6.2}$$

Lemma 6.7. *There are m irreducible characters of D_{2m+1} of degree 2, $\chi_1, \ldots, \chi_m$, and two irreducible characters χ_{m+1}, χ_{m+2} of degree 1. The character table is shown below.*

	β^r	$\alpha\beta^r$
$\chi_j \quad (1 \leq j \leq m)$	$\omega^{rj} + \omega^{-rj}$	0
χ_{m+1}	1	-1
χ_{m+2}	1	1

(where $\omega = \exp(2\pi \mathbf{i}/(2m+1))$*)*

Proof. Ledermann [**Le**, page 66]. ∎

We are now able to completely determine the spectrum of $D_{2m+1}(\Omega_{2m+1})$, using Theorem 6.3.

Theorem 6.8.

$$\operatorname{spec}(D_{2m+1}(\Omega_{2m+1})) = \{\, \lambda_{j1}, \lambda_{j2} \mid 1 \leq j \leq m \,\} \cup \{\, \lambda_{m+1}, \lambda_{m+2} \,\},$$

where

$$\lambda_{j1}, \lambda_{j2} = \sum_{i=0}^{2m} \tau_i \cos \frac{2\pi ji}{2m+1}$$

$$\pm \left(\sum_{i,l=0}^{2m} \tau_i \tau_l \cos \frac{2\pi j(i+l)}{2m+1} + \sum_{i,l=0}^{2m} \kappa_i \kappa_l \cos \frac{2\pi j(i-l)}{2m+1} - \left(\sum_{i=0}^{2m} \tau_i \cos \frac{2\pi ji}{2m+1} \right)^2 \right)^{\frac{1}{2}}$$

and

$$\lambda_{m+1} = 2s_m - k$$

$$\lambda_{m+2} = k.$$

Proof. By Theorem 6.3 and Lemma 6.7 there is an eigenvalue λ_{m+2} of multiplicity one given by

$$\lambda_{m+2} = \sum_{\omega \in \Omega_{2m+1}} \chi_{m+2}(\omega)$$

$$= k.$$

Similarly there is an eigenvalue λ_{m+1} of multiplicity one given by

$$\lambda_{m+1} = \sum_{\omega \in \Omega_{2m+1}} \chi_{m+1}(\omega)$$

$$= s_m - (k - s_m)$$

$$= 2s_m - k.$$

Finally, for $1 \leq j \leq m$, there are eigenvalues λ_{j1}, λ_{j2} of multiplicity two satisfying

$$\lambda_{j1} + \lambda_{j2} = \sum_{\omega \in \Omega_{2m+1}} \chi_j(\omega)$$

$$\lambda_{j1}^2 + \lambda_{j2}^2 = \sum_{\omega, \omega' \in \Omega_{2m+1}} \chi_j(\omega\omega').$$

We split up the latter sum into 4 smaller sums, depending on whether ω (and ω') is a rotation or a reflection. If we set $\Omega_{2m+1} = A \cup B$, where A and B are the set of rotations and reflections in Ω_{2m+1} respectively, this gives, by Lemma 6.6 (ii), (iii) and Lemma 6.7

$$\lambda_{j1}^2 + \lambda_{j2}^2 = \sum_{\omega, \omega' \in A} \chi_j(\omega\omega') + \sum_{\omega, \omega' \in B} \chi_j(\omega\omega')$$

$$= 2\sum_{i=0}^{2m}\sum_{l=0}^{2m} \tau_i \tau_l \cos\frac{2\pi(i+l)j}{2m+1} + 2\sum_{i=0}^{2m}\sum_{l=0}^{2m} \kappa_i \kappa_l \cos\frac{2\pi(i-l)j}{2m+1}. \tag{6.3}$$

We also have

$$\lambda_{j1} + \lambda_{j2} = \sum_{\omega \in A} \chi_j(\omega)$$

$$= 2\sum_{i=0}^{2m} \tau_i \cos\frac{2\pi j i}{2m+1}. \tag{6.4}$$

We can solve for λ_{j1}, λ_{j2} from (6.3) and (6.4) to obtain the required result. ∎

Theorem 6.9. *Suppose that $|\Omega_{2m+1}| = k$ independently of m, so that $D_{2m+1}(\Omega_{2m+1})$ is a k-regular graph of order $4m+2$. Then $\lambda_1(G_{2m+1}(\Omega_{2m+1})) \to k$ as m tends to infinity.*

Proof. From Theorem 6.8 we have $\lambda_{m+2} = k$, so that

$$\lambda_1(G_{2m+1}(\Omega_{2m+1})) \geq \max_{1 \leq j \leq m} \lambda_{j1}.$$

We define the multiset F_{2m+1} of integers modulo $2m+1$ by

$$F_{2m+1} = \{\, i \mid \text{with multiplicity} \quad \tau_i, \quad 0 \leq i \leq 2m \,\}$$

$$\cup \{\, i + l \mid \text{with multiplicity} \quad \tau_i \tau_l, \quad 0 \leq i, l \leq 2m \,\}$$

$$\cup \{\, i - l \mid \text{with multiplicity} \quad \kappa_i \kappa_l, \quad 0 \leq i, l \leq 2m \,\}.$$

Then F_{2m+1} consists of the arguments x of $\cos(2\pi jx/(2m+1))$ that appear in the expression for λ_{j1} in the statement of Theorem 6.8. Clearly $|F_{2m+1}| = s_m + s_m^2 + (k - s_m)^2$ which is bounded above (by $2k^2 + k$) for all m. Hence we may apply Theorem 5.8 to this sequence of multisets. We fix a constant $0 < \epsilon < 1$. Then there is an integer $N = N(\epsilon)$ such that for any $m \geq N$, there is a $t(m) \in \{1, 2, \ldots, 2m\}$ with the property that

$$\cos \frac{2\pi \alpha t(m)}{2m+1} \geq 1 - \epsilon \qquad \forall \alpha \in F_{2m+1}.$$

As before, we may insist that $t(m) \in \{1, 2, \ldots, m\}$. Suppose that $s_m = k$. Then it follows from the formula for $\lambda_{t(m),1}$ given in Theorem 6.8 that

$$k \geq \lambda_1(G_{2m+1}(\Omega_{2m+1})) \geq \lambda_{t(m),1} \geq \sum_{i=0}^{2m} \tau_i \cos \frac{2\pi t(m) i}{2m+1}$$

$$\geq (1 - \epsilon) \sum_{i=0}^{2m} \tau_i$$

$$= k(1 - \epsilon).$$

Thus, if we let $\epsilon \to 0$ by letting $m \to \infty$, it follows that $\lambda_1(G_{2m+1}(\Omega_{2m+1})) \to k$ as $m \to \infty$. For this reason we may assume, without loss of generality, that $s_m \leq k-1$. Then, for sufficiently small ϵ (for instance, $\epsilon < (1+(k-1)^2)^{-1}$ will do), Theorem 6.8 implies that

$$\lambda_{t(m),1} \geq s_m(1-\epsilon) + \left(s_m^2(1-\epsilon) + (k-s_m)^2(1-\epsilon) - s_m^2\right)^{\frac{1}{2}}$$

$$= s_m(1-\epsilon) + (k-s_m)\left(1 - \left(1 + \left(\frac{s_m}{k-s_m}\right)^2\right)\epsilon\right)^{\frac{1}{2}}$$

$$\geq s_m(1-\epsilon) + (k-s_m)\Big(1 - (1+(k-1)^2)\epsilon\Big)^{\frac{1}{2}}$$

$$\geq k\Big(1 - (1+(k-1)^2)\epsilon\Big)^{\frac{1}{2}}$$

which tends to k as ϵ tends to 0. Hence, since we have $\lambda_1(G_{2m+1}(\Omega_{2m+1})) \geq \lambda_{t(m),1}$, then it follows that $\lambda_1(G_{2m+1}(\Omega_{2m+1})) \to k$ as $m \to \infty$, as required. ∎

6.4 The Derived Subgroup Lemma

If G is a group, the *commutator* of $x, y \in G$ is defined to be

$$[x,y] = x^{-1}y^{-1}xy.$$

Then $G' = \langle\, [x,y] \mid x, y \in G \,\rangle$ is called the *derived subgroup* of G, and G/G' is Abelian. Consequently, all irreducible characters of G/G' are linear and, as the next lemma makes clear, they 'supply' all the linear characters of G.

Lemma 6.10. *The group G has $|G:G'|$ linear characters. There is a bijection between the irreducible (and necessarily linear) characters λ_0 of G/G' and the linear (and hence irreducible) characters λ of G, given by*

$$\lambda(g) = \lambda_0(G'g) \qquad \forall g \in G.$$

Proof. Ledermann [**Le**, Theorem 2.8]. ▪

Theorem 6.11. *Suppose* $\{G_n(\Omega_n)\}_{n=1}^{\infty}$ *is a sequence of k-regular Cayley graphs whose orders tend to infinity with n. Then if* $\limsup_{n\to\infty} |G_n : G'_n| = \infty$, *it follows that* $\limsup_{n\to\infty} \lambda_1(G_n(\Omega_n)) = k$.

Proof. Let $|G_n : G'_n| = \alpha(n)$, and suppose that $\phi_0^{(1)}, \ldots, \phi_0^{(\alpha(n))}$ are the distinct irreducible (and hence linear) characters of G_n/G'_n. Let $\phi^{(1)}, \ldots, \phi^{(\alpha(n))}$ be the corresponding linear characters of G_n supplied by Lemma 6.10. Then by Theorem 6.3, for each $1 \le i \le \alpha(n)$, $G_n(\Omega_n)$ has an eigenvalue

$$\lambda_i = \sum_{x\in\Omega_n} \phi^{(i)}(x)$$

$$= \sum_{G'_n x \in \Omega_n/G'_n} \phi_0^{(i)}(G'_n x),$$

(where $\Omega_n/G'_n = \{ G'_n x \mid x \in \Omega_n \}$ is a multiset of size k), which is by Theorem 6.3 an eigenvalue of the Cayley graph $\Gamma_n = (G_n/G'_n)(\Omega_n/G'_n)$. It follows that the spectrum of Γ_n is entirely contained in that of $G_n(\Omega_n)$. But Γ_n is a Cayley graph of an Abelian group and the order of Γ_n is equal to $\alpha(n)$, which satisfies $\limsup_{n\to\infty} \alpha(n) = \infty$. Hence there exists a subsequence $\{\Gamma_{n_r}\}_{r=1}^{\infty}$ of graphs whose orders tend to infinity with r, so that $\lim_{r\to\infty} \lambda_1(\Gamma_{n_r}) = k$ by Theorem 6.5. Hence, because

$$k \ge \lambda_1(G_{n_r}(\Omega_{n_r})) \ge \lambda_1(\Gamma_{n_r}),$$

then it follows that $\limsup_{n\to\infty} \lambda_1(G_n(\Omega_n)) = k$, as required. ▪

We can use this result to prove that several classes of Cayley graphs can never make families of linear enlargers. To begin with, we will supply a fairly elementary example, followed by one of more interest, but first note the following.

Remark. The converse of Theorem 6.11 does not hold in general; that is, there exist sequences of k-regular Cayley graphs based on groups $G_1, G_2, \ldots$, whose subdominant eigenvalues tend to k as their orders tend to infinity, but are such that the orders of G_n/G'_n are bounded for all n. An easy example is the family of Cayley graphs of Dihedral groups defined in the previous section. As was proved in that section, these do not supply families of linear enlargers, but, if α and β are the generators of D_{2m+1} as defined before, it is easy to see that

$$D'_{2m+1} = \langle \beta \rangle,$$

so that $|D_{2m+1} : D'_{2m+1}| = 2$ for all m. ∎

The first family of Cayley graphs which we shall investigate by using Theorem 6.11 are those of the *extra-special p-groups*. If a group P is an extra-special p-group, then its order is a power of a prime p, and either P is elementary Abelian or $|P'| = p$.

Theorem 6.12. *Let $\{P_n(\Omega_n)\}_{n=1}^{\infty}$ be a sequence of k-regular Cayley graphs of extra-special p_n-groups, where P_n has order $p_n^{e_n}$. Then, if $|P_n| \to \infty$ with n, we have*

$$\lim_{n\to\infty} \lambda_1(P_n(\Omega_n)) = k.$$

Proof. We may assume without loss of generality, by the result of Theorem 6.5, that none of the P_n are Abelian. Then it follows that $|P'_n| = p_n$, so that $|P_n : P'_n| = p_n^{e_n - 1}$. But $e_n \geq 2$, for otherwise P_n is Abelian (this is because the extra-special p_n-group P_n has centre $Z(P_n) = P'_n$), so that $|P_n : P'_n| \to \infty$ as $n \to \infty$. The result then follows from Theorem 6.11. ∎

We now apply Theorem 6.11 to Cayley graphs of certain matrix groups.

Definition 6.13. *Let R_n be a ring with unity, of cardinality n. Then for any integer $m \geq 2$ define*

$$T_m(R_n) = \{\, [a_{ij}] \in GL_m(R_n) \mid a_{ii} = 1 \quad \forall i, a_{ij} = 0 \quad if \quad i > j \,\},$$

where $GL_m(R_n)$ is the group of $m \times m$ invertible matrices over R_n.

It is straightforward to check that $T_m(R_n)$ is a group under ordinary matrix multiplication, and we now prove that such groups can never supply families of linear enlargers.

Theorem 6.14. *Let $\{G_r(\Omega_r)\}_{r=1}^{\infty}$ be a sequence of k-regular Cayley graphs, where:*

(a) For all r, $G_r = T_{m_r}(R_{n_r})$ for some integers $m_r, n_r \geq 2$ and some ring with unity R_{n_r} of cardinality n_r;

(b) $|G_r| \to \infty$ as $r \to \infty$.

Then $\lambda_1(G_r(\Omega_r)) \to k$ as $r \to \infty$.

Proof. Let $A \in T_m(R_n)$ be given by

$$A = \begin{pmatrix} 1 & a_{12} & * & \cdots & * & * \\ 0 & 1 & a_{23} & \cdots & * & * \\ 0 & 0 & 1 & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & a_{m-1,m} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

so that

$$A^{-1} = \begin{pmatrix} 1 & -a_{12} & * & \cdots & * & * \\ 0 & 1 & -a_{23} & \cdots & * & * \\ 0 & 0 & 1 & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -a_{m-1,m} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Hence, if $B = [b_{ij}] \in T_m(R_n)$, then

$$AB = \begin{pmatrix} 1 & b_{12} + a_{12} & * & \cdots & * & * \\ 0 & 1 & b_{23} + a_{23} & \cdots & * & * \\ 0 & 0 & 1 & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & b_{m-1,m} + a_{m-1,m} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

and a simple calculation establishes that

$$[A, B] = \begin{pmatrix} 1 & 0 & * & \cdots & * & * \\ 0 & 1 & 0 & \cdots & * & * \\ 0 & 0 & 1 & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Furthermore, any two elements in $T_m(R_n)$ of the above form will multiply to give another of the same form, and the form is also preserved under the taking of inverses. Thus, for any $X = [x_{ij}] \in T'_m(R_n)$ (the derived subgroup), we have $x_{i,i+1} = 0$ for $1 \leq i \leq m-1$. Thus

$$|T'_m(R_n)| \leq n^{\frac{1}{2}(m-1)(m-2)},$$

while it is clear that

$$|T_m(R_n)| = n^{\frac{1}{2}m(m-1)}.$$

Hence

$$|T_{m_r}(R_{n_r}) : T'_{m_r}(R_{n_r})| \geq n_r^{m_r-1}.$$

We know that $|T_{m_r}(R_{n_r})| \to \infty$ with r, and since $m_r \geq 2$ for all r, this clearly implies that $n_r^{m_r-1} \to \infty$ with r. Thus $\lambda_1(G_r(\Omega_r)) \to k$ as $r \to \infty$, by Theorem 6.11, and this completes the proof. ∎

Chapter Seven

Bounding the Gap for Vertex-Transitive Graphs

7.1 Vertex-Transitive Graphs

Let $G = (V, E)$ be a simple graph. An *automorphism* of G is a bijection $\phi: V \to V$ such that $\{u, v\} \in E$ if and only if $\{\phi(u), \phi(v)\} \in E$. The set of all automorphisms of G forms a group $\mathrm{Aut}(G)$ under composition which acts in the obvious way on the set V. If this action is transitive then G is said to be *vertex-transitive*. In this section we will derive an upper bound for the 'gap' ϵ of a vertex-transitive graph which is an (n, k, ϵ)-enlarger.

Let Γ be a finite group, Γ' its derived subgroup. Then Γ/Γ' is Abelian and, by virtue of Theorem 6.1, we may write

$$\Gamma/\Gamma' \cong \mathbf{Z}_{n_1} \oplus \cdots \oplus \mathbf{Z}_{n_r},$$

a direct product of cyclic subgroups whose orders satisfy $n_1 | n_2 | \cdots | n_r$ (the torsion invariants of Γ/Γ').

Lemma 7.1. *The set of (necessarily) linear characters of an Abelian group H forms a multiplicative group isomorphic to H, the isomorphism being defined by the correspondence given in Theorem 6.2.*

Proof. Ledermann [Le, Theorem 2.4]. ▪

We know, from Lemma 6.10, that Γ has precisely $|\Gamma : \Gamma'|$ distinct complex linear characters; a character λ_0 of Γ/Γ' lifts to the linear character λ of Γ given by

$$\lambda(\gamma) = \lambda_0(\Gamma'\gamma) \qquad \forall \gamma \in \Gamma,$$

and all linear characters of Γ are supplied in this way. For an arbitrary finite group H the *exponent* $\theta = \theta(H)$ of H is the least common multiple of the orders of the elements of H. In particular, because Γ/Γ' is Abelian, its exponent $\theta(\Gamma/\Gamma') = n_r$, the largest torsion invariant of Γ/Γ'.

Clearly, there must be an element of Γ/Γ' of order $\theta = \theta(\Gamma/\Gamma')$ which yields, by Lemma 7.1, a linear character ϕ_0 of Γ/Γ' of order θ. Then ϕ_0 lifts in the manner described above to a linear character ϕ of Γ. The order of ϕ_0 being θ, there must exist $\gamma \in \Gamma$ for which $\phi_0(\Gamma'\gamma)$ (and hence $\phi(\gamma)$) is a primitive root of $x^\theta - 1$, so that the order of ϕ is θ too. Because ϕ is a linear character of order θ, it follows that there exists a function $\hat{\phi}\colon \Gamma \to \mathbf{Z}_\theta$ given by

$$\phi(\gamma) = \exp\left(\frac{2\pi \mathrm{i}\hat{\phi}(\gamma)}{\theta}\right),$$

where, because ϕ is a linear character,

(a) $\hat{\phi}(\gamma) = 0$ for any $\gamma \in \Gamma'$;

(b) $\hat{\phi}(\gamma_1\gamma_2) = \hat{\phi}(\gamma_1) + \hat{\phi}(\gamma_2)$ for any $\gamma_1, \gamma_2 \in \Gamma$.

Theorem 7.2. *Let $\Gamma(\Omega)$ be a k-regular Cayley graph, $\theta = \theta(\Gamma/\Gamma')$ and $\hat{\phi}$ defined as above. Then*

$$\lambda_j = \sum_{\omega\in\Omega} \cos\frac{2\pi\hat{\phi}(\omega)j}{\theta}$$

is an eigenvalue of $\Gamma(\Omega)$, for $j = 0, 1, \ldots, \theta - 1$.

Proof. Since ϕ is a linear character of order θ, then clearly $1, \phi, \ldots, \phi^{\theta-1}$ are distinct linear characters of Γ. The result then follows from an application of Theorem 6.3. ∎

We will need the following Theorem of Klawe.

Lemma 7.3. *Let $G_m(\Psi_m)$ be a k-regular Klawe graph of order $2m$ (see Chapter 5 for a definition), where $\Psi_m = \{ f_1, \ldots, f_k \}$ and f_1 is the identity transformation. Then, for any fixed $\alpha \in (0,1)$ there is a subset X of the inputs of $G_m(\Psi_m)$ satisfying*

(i) $\alpha m/2 \le |X| \le \alpha m$, and

(ii)

$$|f_i(X) \setminus f_1(X)| < \frac{3|X|}{\lfloor \alpha(\log \alpha m / \log\log \alpha m)^{1/(3k+2)} \rfloor},$$

where $\log x = \log_2 x$, *and provided that* $\log\log \alpha m > 0$ *(that is,* $m > 2/\alpha$*).*

Proof. This is a direct consequence of a result of Klawe [**Kl**, Theorem 2.13]. ∎

Corollary 7.4. *If* $G_m(\Psi_m)$ *is the Klawe graph given in Lemma 7.3, and it is an* (m, k, d)*-expander, then, provided* $m > 4$,

$$d < \frac{6(k-1)}{\left\lfloor \frac{1}{2}\left(\log \frac{m}{2} / \log\log \frac{m}{2}\right)^{1/(3k+2)} \right\rfloor}.$$

Proof. If $m > 4$ then $\alpha = \frac{1}{2}$ satisfies $m > 2/\alpha$. Therefore, by Lemma 7.3 there is a subset X of inputs of $G_m(\Psi_m)$ with $m/4 \le |X| \le m/2$ and

$$|\Gamma(X)| \le |f_1(X)| + \sum_{i=2}^{k} |f_i(X) \setminus f_1(X)|$$

$$< |X| + \frac{3(k-1)|X|}{\left\lfloor \frac{1}{2}\left(\log \frac{m}{2} / \log\log \frac{m}{2}\right)^{1/(3k+2)} \right\rfloor}.$$

(Here, $\Gamma(X)$ denotes the set of neighbours of X in the graph $G_m(\Psi_m)$, as opposed to the group Γ. It will always be clear from the context which interpretation is intended.) Also, $|X| \le m/2$ implies that

$$|\Gamma(X)| \ge \left(1 + d\left(1 - \frac{|X|}{m}\right)\right)|X|$$

$$\ge \left(1 + \frac{d}{2}\right)|X|.$$

Hence

$$1 + \frac{3(k-1)}{\left\lfloor \frac{1}{2}\left(\log \frac{m}{2} / \log\log \frac{m}{2}\right)^{1/(3k+2)} \right\rfloor} > 1 + \frac{d}{2},$$

which leads to the result required. ■

From Corollary 7.4 we can obtain an upper bound for the 'gap' ϵ of a Cayley graph $\Gamma(\Omega)$ in terms of the exponent of Γ/Γ'.

Theorem 7.5. *Let $\Gamma(\Omega)$ be a k-regular Cayley graph which is also an (n,k,ϵ)-enlarger. Let θ be the exponent of Γ/Γ', and define the function*

$$f_k(\theta) = \frac{12k}{\left\lfloor \frac{1}{2}\left(\log\frac{\theta}{2}/\log\log\frac{\theta}{2}\right)^{1/(6k+5)}\right\rfloor}.$$

Then, provided $\theta > 4$ and $f_k(\theta) < 1$, we have

$$\epsilon < \frac{k f_k(\theta)}{2(1 - f_k(\theta))}.$$

Proof. Let the function $\hat{\phi}$ be as defined before. Then, from Theorem 5.5 applied to the multiset of k integers $\{\,\hat{\phi}(\omega) \mid \omega \in \Omega\,\}$, chosen from $\mathbf{Z}_\theta$, we deduce the existence of a circulant graph Δ of order θ and valency $2k$ with

$$\operatorname{spec}(\Delta) = \{\, 2\lambda_j \mid j = 0, 1, \ldots, \theta - 1\,\},$$

where the λ_j are as given in Theorem 7.2.

The augmented double cover $\tilde{\Delta}$ of Δ is then a Klawe graph of order 2θ and valency $2k+1$ (by Theorem 5.6). Thus, since $\theta > 4$, its expansion d satisfies

$$\begin{aligned} d &< \frac{12k}{\left\lfloor \frac{1}{2}\left(\log\frac{\theta}{2}/\log\log\frac{\theta}{2}\right)^{1/(6k+5)}\right\rfloor} \\ &= f_k(\theta) \qquad \text{by definition.} \end{aligned}$$

Then, from Lemmas 1.5 and 1.3(i) we deduce that, if Δ is a $(\theta, 2k, \tilde{\epsilon})$-enlarger, then

$$\frac{\tilde{\epsilon}}{k+\tilde{\epsilon}} < f_k(\theta).$$

That is,

$$\tilde{\epsilon} < \frac{k f_k(\theta)}{1 - f_k(\theta)},$$

provided that $f_k(\theta) < 1$.

Also, $\tilde{\epsilon}$ satisfies

$$\begin{aligned}\tilde{\epsilon} &= 2k - \lambda_1(\Delta)\\ &= 2k - 2\lambda_j \qquad \text{for some } j \in \{1, 2, \ldots, \theta - 1\}\\ &\geq 2k - 2\lambda_1(\Gamma(\Omega)),\end{aligned}$$

because each λ_j $(j = 1, 2, \ldots, \theta - 1)$ is an eigenvalue of $\Gamma(\Omega)$ not greater than $\lambda_1(\Gamma(\Omega))$. Hence, if $\Gamma(\Omega)$ is an (n, k, ϵ)-enlarger, then $\tilde{\epsilon} \geq 2\epsilon$ so that

$$\epsilon < \frac{k f_k(\theta)}{2(1 - f_k(\theta))}$$

as required. ∎

In practice, it can be seen that the condition $f_k(\theta) < 1$ is much more restrictive than $\theta > 4$, for it requires that

$$12k < \left\lfloor \frac{1}{2}\left(\log\frac{\theta}{2} / \log\log\frac{\theta}{2}\right)^{1/(6k+5)} \right\rfloor,$$

which forces θ to be very large.

Cayley graphs are a special case of vertex-transitive graphs, and we can extend the above result to this larger class of graphs as follows: let G be a k-regular vertex-transitive graph of order n with automorphism group $\Gamma = \mathrm{Aut}(G)$. We define a new graph G_C by $V(G_C) = \Gamma$, and $\{\gamma_1, \gamma_2\} \in E(G_C)$ if and only if $\{\gamma_1(x), \gamma_2(x)\} \in E(G)$, where x is a fixed vertex of G. Then it follows that G_C is a Cayley graph $\Gamma(\Omega)$, where

$$\Omega = \{\, \gamma \in \Gamma \mid \{x, \gamma(x)\} \in E(G) \quad \text{for some fixed} \quad x \in V(G)\,\}.$$

The graph G_C is of order $|\Gamma|$ and valency ks, where $s = |\Gamma|/|G|$. The spectrum of $\Gamma(\Omega)$ is easily determined from that of G.

Theorem 7.6. *Let G, Γ, Ω, s, k and n be as given above. Then, if we remove $(s-1)n$ zeros from* $\mathrm{spec}(\Gamma(\Omega))$ *and divide each remaining element by s, we are left with* $\mathrm{spec}(G)$.

Proof. Lovász [**Lo**]. ∎

Corollary 7.7. *Let G be a k-regular graph of order n with transitive automorphism group Γ, and let $s = |\Gamma|/|G|$. Let θ be the exponent of Γ/Γ'. If G is an (n,k,ϵ)-enlarger and we define*

$$f_{ks}(\theta) = \frac{12ks}{\left\lfloor \frac{1}{2}\left(\log\frac{\theta}{2}/\log\log\frac{\theta}{2}\right)^{1/(6ks+5)}\right\rfloor},$$

then

$$\epsilon < \frac{kf_{ks}(\theta)}{2\big(1-f_{ks}(\theta)\big)},$$

provided that $\theta > 4$ and $f_{ks}(\theta) < 1$.

Proof. We form the Cayley graph $\Gamma(\Omega)$ from G in the manner just described. Then $\Gamma(\Omega)$ is ks-regular. Now it is true that $\lambda_1 > 0$ for any connected graph that is not a complete multipartite graph (see Smith [**Sm**] for a proof). However, if the graph is complete multipartite (with multipartite blocks $B_1, \ldots, B_r$) and is also ks-regular of order $|\Gamma|$, then

$$\sum_{j\neq i} |B_j| = ks,$$

$$\sum_{j=1}^{r} |B_j| = \frac{ks|\Gamma|}{2}.$$

Hence $|B_j| = ks(|\Gamma|-2)/2$ for all j, so that $rks(|\Gamma|-2)/2 = ks|\Gamma|/2$, which implies that $|\Gamma| \le 4$. Consequently, if the exponent of Γ/Γ' is at least 5, then the order of Γ/Γ' (and hence that of Γ) is at least 5 and so $\Gamma(\Omega)$ cannot be complete multipartite. We deduce that $\lambda_1(\Gamma(\Omega)) > 0$. It then follows from Theorem 7.6 that

$$\lambda_1(G) = \frac{\lambda_1(\Gamma(\Omega))}{s}.$$

As a result of this,

$$k - \lambda_1(G) = k - \frac{\lambda_1(\Gamma(\Omega))}{s}$$

$$= \frac{ks - \lambda_1(\Gamma(\Omega))}{s},$$

so that, if $\Gamma(\Omega)$ is an $(|\Gamma|, ks, \tilde{\epsilon})$-enlarger, then G is a (n, k, ϵ)-enlarger and

$$\epsilon = \frac{\tilde{\epsilon}}{s}.$$

Using Theorem 7.5 applied to Γ with $\theta = \theta(\Gamma/\Gamma')$ we deduce that, provided $\theta > 4$ and $f_{ks}(\theta) < 1$,

$$\epsilon < \frac{ksf_{ks}(\theta)}{2s(1 - f_{ks}(\theta))}$$

whence the result.∎

Note that, for $s = 1$, this result reduces to that of Theorem 7.5, for then $|\Gamma| = |G|$, so that Γ acts regularly on G, and G is itself a Cayley graph.

7.2 Ramanujan Graphs

Suppose that $\{G_n\}_{n=1}^{\infty}$ is a family of connected k-regular graphs whose orders tend to infinity with n. Define the parameter

$$\Lambda(G_n) = \max\{\,|\lambda| \mid \lambda \in \mathrm{spec}(G_n) \setminus \{k, -k\}\,\}.$$

There is a simple asymptotic lower bound on the value of $\Lambda(G_n)$ as n tends to infinity, due to Alon and Boppana. We include another proof of this bound, due to Lubotzky, Phillips and Sarnak [**LPS**].

Theorem 7.8.

$$\liminf_{n\to\infty} \Lambda(G_n) \geq 2\sqrt{k-1}.$$

Proof. Let G be a k-regular graph of order n, with adjacency matrix A, so that $A^l = [a_{ij}^{(l)}]$ where

$$a_{ij}^{(l)} = \#\text{ walks between vertices } i \text{ and } j \text{ of length } l \text{ in } G.$$

Suppose $\lambda_0 = k \geq \lambda_1 \geq \cdots \geq \lambda_{n-1}$ are the eigenvalues of G. Then, since $\mathrm{Tr}(A^l) = \sum_{j=0}^{n-1} \lambda_j^l$ it follows that

$$\sum_{j=0}^{n-1} \lambda_j^l = \sum_{j=0}^{n-1} a_{jj}^{(l)}.$$

We define a *reduced walk* in G to be a sequence $v_1 v_2 \ldots v_r$ of vertices of G such that $\{v_i, v_{i+1}\} \in EG$ for $1 \leq i \leq r-1$ and $v_i \neq v_{i+2}$ for $1 \leq i \leq r-2$. The *universal cover* $U(G,v)$ of G (with respect to a fixed vertex v) is the infinite graph whose vertices are the reduced walks in G with origin v, two such walks being adjacent if and only if one is a single-step extension of the other. Clearly $U(G,v) \cong T^k$, the infinite k-regular tree, so that $a_{jj}^{(l)} \geq \rho(l)$, where $\rho(l)$ is the number of walks of length l from any vertex x to itself in T^k. Thus

$$\sum_{j=0}^{n-1} \lambda_j^l \geq n\rho(l),$$

so that, after removing (if possible) k and $-k$ from the spectrum,

$$\Lambda(G)^{2l} \geq \rho(2l) - \frac{2k^{2l}}{n-2}. \tag{7.1}$$

Obviously, $\rho(2l) \geq \rho'(2l)$, the number of walks of length $2l$ with origin x that end on the first return to x in T^k, and it can be shown that

$$\rho'(2l) = \frac{1}{l}\binom{2l-2}{l-1} k(k-1)^{l-1}. \tag{7.2}$$

Thus (7.1) and (7.2) give

$$\Lambda(G)^{2l} \geq \frac{1}{l}\binom{2l-2}{l-1}(k-1)^l - \frac{2k^{2l}}{n-2}.$$

However, if a, b and c are non-negative, then $a^{2l}+b^{2l} \geq c^{2l}$ implies that $a+b \geq c$, so that

$$\Lambda(G) \geq \left(\frac{1}{l}\right)^{\frac{1}{2l}} \binom{2l-2}{l-1}^{\frac{1}{2l}} \sqrt{k-1} - \frac{2^{1/2l}k}{(n-2)^{1/2l}}.$$

Now

$$\lim_{l\to\infty} \binom{2l-2}{l-1}^{\frac{1}{2l}} \left(\frac{1}{l}\right)^{\frac{1}{2l}} = 2$$

so that, for any $\epsilon > 0$, there exists $l_0 = l_0(\epsilon)$ such that

$$\Lambda(G) \geq 2\sqrt{k-1} - \frac{\epsilon}{2} - \frac{2^{1/2l}k}{(n-2)^{1/2l}} \qquad \forall l \geq l_0.$$

Given l, let $n_0 = n_0(l, \epsilon)$ be such that

$$\frac{2^{1/2l}k}{(n-2)^{1/2l}} < \frac{\epsilon}{2} \qquad \forall n \geq n_0.$$

It then follows that, for any $\epsilon > 0$, there is an $n_0 = n_0(\epsilon)$ such that

$$\Lambda(G) > 2\sqrt{k-1} - \epsilon \qquad \forall n \geq n_0,$$

and the result follows. ∎

A k-regular graph G satisfying $\Lambda(G) \leq 2\sqrt{k-1}$ is known as a *Ramanujan graph*, because the first construction of a linear family $\{G_r\}_{r=1}^{\infty}$ of such graphs [**LPS**] used a special case of Ramanujan's conjecture [**Ra**] (that was proved by Eichler [**Ei**]) to demonstrate the fact that $\lim_{r\to\infty} \Lambda(G_r) = 2\sqrt{k-1}$.

Although the characterisation of Ramanujan graphs is a spectral one, it is possible to say something about their magnifying properties. We can use the results of Lemma 1.3(i) and Corollary 2.2 to obtain the next result.

Theorem 7.9. *Let $\{G_r\}_{r=1}^{\infty}$ be a family of k-regular Ramanujan graphs with i_r being the isoperimetric number of G_r. Then*

$$\left(1+\frac{k}{2(k-2\sqrt{k-1})}\right)^{-1} \leq \liminf_{r\to\infty} i_r \leq \limsup_{r\to\infty} i_r \leq 1-2^{-k}. \blacksquare$$

Ramanujan graphs are important because of their good enlarging properties, as is evident from the bound of Theorem 7.8. (Indeed, the family of such graphs constructed in [**LPS**] gave superconcentrators of density 58, the lowest yet achieved.) In fact, the largest possible 'gap' for any family of *bipartite* k-regular enlargers is $k-2\sqrt{k-1}$, since $\Lambda(G)=\lambda_1(G)$ if G is bipartite, and such a family must necessarily consist of Ramanujan graphs.

It is clear from Theorem 6.11 that a family of k-regular Ramanujan Cayley graphs $\{\Gamma_n(\Omega_n)\}_{n=1}^{\infty}$ must be such that there is a positive constant M with $|\Gamma_n:\Gamma_n'|<M$ for all n. The next theorem uses Theorem 7.5 to provide an upper bound on the exponent of Γ/Γ' for the Cayley graph $\Gamma(\Omega)$ to be Ramanujan.

Theorem 7.10. *Let $k\geq 3$ be an integer. Then if $\Gamma(\Omega)$ is a k-regular Ramanujan Cayley graph, the exponent $\theta=\theta(\Gamma/\Gamma')$ must satisfy $\log\theta<(2^{\alpha_k}+1)$ where $\log\theta=\log_2\theta$ and*

$$\alpha_k=\left(2+\frac{12k(3k-4\sqrt{k-1})}{k-2\sqrt{k-1}}\right)^{6k+5}.$$

Proof. Suppose $\theta\geq 5$ and $f_k(\theta)\leq\frac{2k-4\sqrt{k-1}}{3k-4\sqrt{k-1}}$. Then, by Theorem 7.5,

$$\epsilon<\frac{kf_k(\theta)}{2(1-f_k(\theta))}$$

$$\leq\frac{k}{2}\left(\frac{3k-4\sqrt{k-1}}{k}-1\right)$$

$$=k-2\sqrt{k-1},$$

so that $\Gamma(\Omega)$ cannot be Ramanujan. Hence, if $\Gamma(\Omega)$ is Ramanujan, we require

$$\frac{12k}{\left\lfloor \frac{1}{2}\left(\log\frac{\theta}{2}/\log\log\frac{\theta}{2}\right)^{1/(6k+5)}\right\rfloor} > \frac{2k-4\sqrt{k-1}}{3k-4\sqrt{k-1}}$$

which implies that

$$\alpha_k = \left(2 + \frac{12k(3k-4\sqrt{k-1})}{k-2\sqrt{k-1}}\right)^{6k+5} > \frac{\log\frac{\theta}{2}}{\log\log\frac{\theta}{2}}.$$

Now the function $g(x) = \log(x/2)/\log\log(x/2)$ is monotone increasing for $x \in (8,\infty)$, and

$$\theta = 2^{(2^{\alpha_k}+1)} > 8 \quad \Rightarrow \quad \alpha_k \leq g(\theta),$$

so that, for $\Gamma(\Omega)$ to be Ramanujan, we require that $\log\theta < (2^{\alpha_k}+1)$ which completes the proof. ∎

It is possible to use this bound on the exponent of Γ/Γ' to derive a bound on $|\Gamma : \Gamma'|$, using the next result.

Lemma 7.11. *Let $\Gamma(\Omega)$ be a k-regular connected Cayley graph of order n, and suppose that θ is the exponent of Γ/Γ'. Then*

$$|\Gamma : \Gamma'| \leq \theta^k.$$

Proof. Since Γ has a generating set of size k, then so must Γ/Γ'. Also, every element of Γ/Γ' has order at most θ, and it is an Abelian group, so that

$$|\Gamma : \Gamma'| \leq \theta^k$$

as required. ∎

Theorem 7.12. *Let $\Gamma(\Omega)$ be a k-regular Ramanujan Cayley graph, and define α_k as in the statement of Theorem 7.10. Then*

$$|\Gamma : \Gamma'| < \theta_k^k$$

where $\log \theta_k = 2^{\alpha_k} + 1$.

Proof. If $|\Gamma : \Gamma'| \geq \theta_k^k$ then, since $\Gamma(\Omega)$ is Ramanujan and is thus connected, Lemma 7.11 implies that

$$\theta_k^k \leq |\Gamma : \Gamma'| \leq \theta^k,$$

where θ is the exponent of Γ/Γ'. Hence $\theta \geq \theta_k$ which, by Theorem 7.10, implies that $\Gamma(\Omega)$ cannot be Ramanujan, which is a contradiction. This completes the proof. ∎

In fact we can employ Lemma 7.11 to obtain a bound on the gap of a vertex-transitive graph in terms of the index $|\Gamma : \Gamma'|$ instead of the exponent θ, using Corollary 7.7.

Theorem 7.13. *Let G be a k-regular graph with transitive automorphism group Γ, and let $s = |\Gamma|/|G|$. Then, if we define the function*

$$f(x) = f_{ks}(x^{1/ks})$$

(where f_{ks} is the function defined in Corollary 7.7), and assume that $|\Gamma : \Gamma'| > 2^{3ks}$ and $f(|\Gamma : \Gamma'|) < 1$, it follows that

$$\epsilon < \frac{kf(|\Gamma : \Gamma'|)}{2\big(1 - f(|\Gamma : \Gamma'|)\big)}.$$

Proof. If $\Gamma(\Omega)$ is the ks-regular Cayley graph obtained from G in the manner described in the previous section then as we always assume G to be connected, so is $\Gamma(\Omega)$. Hence, as in the proof of Lemma 7.11, we may deduce that

$$|\Gamma : \Gamma'| \leq \theta^{ks},$$

where θ is the exponent of Γ/Γ'. Now the function $g(x) = \log(x/2)/\log\log(x/2)$ is monotone increasing for $x \in (8, \infty)$, so that the function $f_{ks}(x)$ (as defined in Corollary 7.7) is monotone decreasing in this range. Hence

$$8 < |\Gamma : \Gamma'|^{1/ks} \leq \theta$$

implies that

$$f_{ks}(\theta) \leq f_{ks}(|\Gamma : \Gamma'|^{1/ks})$$

$$= f(|\Gamma : \Gamma'|) \quad \text{by definition.}$$

In other words, $|\Gamma : \Gamma'| > 2^{3ks}$ will ensure that $f_{ks}(\theta) \leq f(|\Gamma : \Gamma'|)$. Notice that it also implies that

$$\theta^{ks} \geq |\Gamma : \Gamma'| > 2^{3ks} > 4^{ks}$$

so that we must have $\theta > 4$. To sum up, if $|\Gamma : \Gamma'| > 2^{3ks}$ and $f(|\Gamma : \Gamma'|) < 1$ then $\theta > 4$ and $f_{ks}(\theta) \leq f(|\Gamma : \Gamma'|) < 1$. The result now follows on applying Corollary 7.7. ∎

Chapter Eight

Criteria for Ramanujan Graphs

8.1 Ramanujan Graphs and Universal Covers

In this chapter we continue from section 7.2 the examination of Ramanujan graphs. We will adopt a different approach from that of Lubotzky, Phillips and Sarnak (who used some quite sophisticated number theory) in the search for such graphs. Given an arbitrary (finite) connected k-regular simple graph, we will consider it as a quotient of its universal cover (see section 7.2 for a definition), which in this case will be isomorphic to the infinite k-regular tree $T^k = (V_k, E_k)$. We define $L^2(T^k)$ to be the space of functions $\mathbf{f} : V_k \to \mathbf{C}$ ($\mathbf{C}$ being the set of complex numbers) which satisfy $\|\mathbf{f}\| < \infty$, where

$$\|\mathbf{f}\|^2 = \sum_{v \in V_k} |\mathbf{f}(v)|^2.$$

Corresponding to the adjacency matrix of a finite graph, an infinite graph $G = (V, E)$ has an adjacency operator A on the space $L^2(G)$ given by

$$A\mathbf{f}(v) = \sum_{\{w,v\} \in E} \mathbf{f}(w)$$

where $\mathbf{f} \in L^2(G)$. The above sum will always have finite modulus in what follows since we will deal with infinite graphs of *finite* valency only. There is a spectrum that can be associated with this operator.

Definition 8.1. *Let $G = (V, E)$ be an infinite graph with adjacency operator A. Suppose $\langle \mathbf{x}_n \rangle_{n=1}^{\infty}$ is a sequence from $L^2(G)$ with $\|\mathbf{x}_n\| = 1$ for all n, and that there is a constant $\lambda \in \mathbf{C}$ such that*

$$\lim_{n \to \infty} \|(\lambda I - A)\mathbf{x}_n\| = 0,$$

where I is the identity operator on $L^2(G)$. The set of all such λ is called the point spectrum $\sigma(G)$ of G. (See Dowson [**Do**].)

The following theorem is fundamental to what follows.

Lemma 8.2. *Let $T^k = (V_k, E_k)$ denote the infinite k-regular tree. Then*

$$\sigma(T^k) = [-2\sqrt{k-1}, 2\sqrt{k-1}\,].$$

Proof. Mohar and Omladič [**MO**], although calculations which lead to the same conclusion were given earlier by MacKay [**McK**] and Kesten [**Ke**]. ∎

Throughout the rest of this chapter $\Gamma = (V, E)$ will denote a finite connected k-regular simple graph with $V = \{v_1, \ldots, v_h\}$. We fix a vertex $v \in V$ and consider Γ as a finite quotient of the universal cover $U(\Gamma, v) \cong T^k$, with quotient map $\phi : V_k \to V$ defined by

$$\phi(vv^{(2)} \ldots v^{(r)}) = v^{(r)}$$

(so that a reduced walk is mapped to its final vertex).

Now suppose that $k = \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{h-1}$ are the eigenvalues of Γ, and that $(\mathbf{f}_j, \lambda_j)$ is an *eigensolution* of Γ (that is, $A\mathbf{f}_j = \lambda_j \mathbf{f}_j$, where A is the adjacency matrix of Γ and $\mathbf{f}_j \in L^2(\Gamma)$ is a real function which satisfies $\|\mathbf{f}_j\| = 1$). We will *lift* $\mathbf{f}_j$ to a sequence $\langle \mathbf{y}_j^{(n)} \rangle_{n=1}^{\infty}$ in $L^2(T^k)$ by fixing the vertex v in V_k and defining

$$\mathbf{y}_j^{(n)}(x) = \begin{cases} \mathbf{f}_j(\phi(x)) & \text{if } \partial(v, x) \leq n \\ 0 & \text{otherwise} \end{cases}$$

where ϕ is the quotient map defined above and ∂ is the usual distance metric on T^k. For future brevity we will write $F_j = \mathbf{f}_j \phi$ and define the subsets of V_k

$$D(r) = \{\, x \in V_k \mid \partial(v, x) = r \,\}$$

$$B(r) = \{\, x \in V_k \mid \partial(v, x) \leq r \,\}$$

for $r = 0, 1, \ldots$. Then

$$\|\mathbf{y}_j^{(n)}\|^2 = \sum_{x \in B(n)} F_j^2(x)$$

$$= \sum_{r=0}^{n} \sum_{x \in D(r)} F_j^2(x). \tag{8.1}$$

For every non-negative integer r, the vertices in $D(r)$ are partitioned into $S_1^{(r)} \cup \ldots \cup S_h^{(r)}$, where

$$S_i^{(r)} = \{ x \in D(r) \mid \phi(x) = v_i \} \qquad (1 \leq i \leq h).$$

In effect, we partition them according to the vertex of Γ that they are mapped to by ϕ. Then, writing $F_j(i)$ for the (constant) value of $F_j(x)$ whenever $\phi(x) = v_i$, we have the following result.

Theorem 8.3. *If $\langle \mathbf{y}_j^{(n)} \rangle_{n=1}^{\infty}$ is the sequence from $L^2(T^k)$ defined above, then*

$$\|\mathbf{y}_j^{(n)}\|^2 = \sum_{r=0}^{n} \phi_r(j) \qquad \forall n \geq 0,$$

where the function $\phi_r : \{1, 2, \ldots, h\} \to \mathbf{R}$ (the set of real numbers) is defined by

$$\phi_r(j) = \sum_{i=1}^{h} |S_i^{(r)}| F_j^2(i).$$

Proof. From (8.1) and the partition of $D(r)$ given above we see that

$$\|\mathbf{y}_j^{(n)}\|^2 = \sum_{r=0}^{n} \sum_{i=1}^{h} |S_i^{(r)}| F_j^2(i)$$

$$= \sum_{r=0}^{n} \phi_r(j),$$

as required. ∎

From the sequence $\langle \mathbf{y}_j^{(n)} \rangle_{n=1}^{\infty}$ we obtain a normalized sequence $\langle \mathbf{Y}_j^{(n)} \rangle_{n=1}^{\infty}$ by defining

$$\mathbf{Y}_j^{(n)} = \begin{cases} \mathbf{y}_j^{(n)}/\|\mathbf{y}_j^{(n)}\| & \text{if } \|\mathbf{y}_j^{(n)}\| \neq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Because $\mathbf{f}_j \neq \mathbf{0}$, there exists i with $F_j^2(i) = \epsilon^2 > 0$. Let n_o be the smallest n for which $S_i^{(n)} \neq \emptyset$. Then, by definition, $\|\mathbf{y}_j^{(n)}\| \geq \epsilon$ so that $\mathbf{Y}_j^{(n)} \neq \mathbf{0}$ for all $n \geq n_0$. Hence we have a sequence $\langle \mathbf{Y}_j^{(n)} \rangle_{n=1}^{\infty}$ with $\|\mathbf{Y}_j^{(n)}\| = 1$ for all $n \geq n_0$. We transform this into another sequence $\langle \mathbf{Z}_j^{(n)} \rangle_{n=1}^{\infty}$ from $L^2(T^k)$ by

$$\mathbf{Z}_j^{(n)} = (A - \lambda_j I)\mathbf{Y}_j^{(n)} \qquad n = 1, 2, \ldots$$

where A is the adjacency operator of T^k.

Lemma 8.4. *For $n \geq 1$,*

$$\mathbf{Z}_j^{(n)}(x) = \begin{cases} 0 & \text{if } \partial(v,x) < n \text{ or } \partial(v,x) > n+1; \\ (\mathbf{Y}_j^{(n)}(x') - \lambda_j \mathbf{Y}_j^{(n)}(x)) & \text{if } \partial(v,x) = n,\ \partial(v,x') = n-1 \text{ and } \partial(x,x') = 1; \\ \mathbf{Y}_j^{(n)}(x') & \text{if } \partial(v,x) = n+1,\ \partial(v,x') = n \text{ and } \partial(x,x') = 1. \end{cases}$$

(Note that the x' given in the last two cases is uniquely defined by x and v, since T^k is a tree.)

Proof. (i) Suppose that $\partial(v,x) < n$. Then

$$\mathbf{Z}_j^{(n)}(x) = A\mathbf{Y}_j^{(n)}(x) - \lambda_j \mathbf{Y}_j^{(n)}(x)$$

$$= \sum_{\partial(x,x')=1} \mathbf{Y}_j^{(n)}(x') - \lambda_j \mathbf{Y}_j^{(n)}(x)$$

$$= \frac{1}{\|\mathbf{y}_j^{(n)}\|}\left(\sum_{\partial(x,x')=1} \mathbf{f}_j(\phi(x')) - \lambda_j \mathbf{f}_j(\phi(x)) \right).$$

Now $\phi : V_k \to V$ is a quotient map, so that the neighbours of $\phi(x)$ in Γ are precisely the images under ϕ of the neighbours of x in T^k. Thus, if $\tilde{\partial}$ is the distance metric on Γ,

$$\mathbf{Z}_j^{(n)}(x) = \frac{1}{\|\mathbf{y}_j^{(n)}\|}\Bigg(\sum_{\tilde{\partial}(u,\phi(x))=1} \mathbf{f}_j(u) - \lambda_j \mathbf{f}_j(\phi(x))\Bigg)$$

$$= 0$$

because $(\mathbf{f}_j, \lambda_j)$ is an eigensolution of Γ.

(ii) Suppose that $\partial(v,x) > n+1$. Then, as in (i)

$$\mathbf{Z}_j^{(n)}(x) = \sum_{\partial(x,x')=1} \mathbf{Y}_j^{(n)}(x') - \lambda_j \mathbf{Y}_j^{(n)}(x).$$

However, if $\partial(v,x) > n+1$ and $\partial(x,x') = 1$, then $\partial(v,x') > n$. Hence, by definition of $\mathbf{y}_j^{(n)}$, we have $\mathbf{Y}_j^{(n)}(x') = \mathbf{y}_j^{(n)}(x') = 0$, and similarly for x. Thus $\mathbf{Z}_j^{(n)}(x) = 0$.

(iii) Suppose that $\partial(v,x) = n$. Then if $\partial(x,x') = 1$, $\partial(v,x') = n+1$ or $n-1$. In the former case we have $\mathbf{Y}_j^{(n)}(x') = 0$, and in the latter case there is a unique such x' (the predecessor of x in the tree T^k rooted at v). Thus

$$\mathbf{Z}_j^{(n)}(x) = \mathbf{Y}_j^{(n)}(x') - \lambda_j \mathbf{Y}_j^{(n)}(x).$$

(iv) Finally, suppose that $\partial(v,x) = n+1$. Then it is clear that

$$\mathbf{Z}_j^{(n)}(x) = \sum_{\partial(x,x')=1} \mathbf{Y}_j^{(n)}(x') - \lambda_j \mathbf{Y}_j^{(n)}(x)$$

$$= \mathbf{Y}_j^{(n)}(x'),$$

where x' is the unique predecessor of x in the tree T^k (rooted at v). ∎

Theorem 8.5. *If $\langle \mathbf{Z}_j^{(n)} \rangle_{n=1}^{\infty}$ is the sequence defined above then*

$$\|\mathbf{Z}_j^{(n)}\|^2 = \frac{(k-1)\sum_{x\in D(n)} F_j^2(x) + \sum_{x\in D(n)} \big(F_j(x') - \lambda_j F_j(x)\big)^2}{\sum_{r=0}^{n} \phi_r(j)} \tag{8.2}$$

where x' is the predecessor of x in the usual sense.

Proof. Immediate from Lemma 8.4 and the fact that, since T^k is a k-regular tree, each vertex of T^k at distance at least one from v is the predecessor of exactly $k-1$ vertices. ∎

We say that the eigensolution $(\mathbf{f}_j, \lambda_j)$ of Γ *lifts* to T^k if the derived sequence $\langle \mathbf{Z}_j^{(n)} \rangle_{n=1}^{\infty}$ satisfies $\liminf_{n\to\infty} \|\mathbf{Z}_j^{(n)}\| = 0$. It is then clear from Definition 8.1 that $\lambda_j \in \sigma(T^k)$, and hence the modulus of λ_j does not exceed $2\sqrt{k-1}$. What we now do is to derive some conditions on the multiplicity of an eigenvalue for it to possess a 'lifting' eigensolution, and examine the implications for the search for Ramanujan graphs.

Corollary 8.6. *The eigensolution $(\mathbf{f}_j, \lambda_j)$ of Γ lifts to T^k if*

$$\liminf_{n\to\infty} \frac{\phi_n(j) + \phi_{n-1}(j)}{\sum_{r=0}^{n} \phi_r(j)} = 0.$$

Proof. Consider equation (8.2) again. If we apply the inequality

$$\sum (a_i - b_i)^2 \leq 2\sum a_i^2 + 2\sum b_i^2$$

between finite series, we get (x' being the predecessor of x)

$$\sum_{x\in D(n)} \left(F_j(x') - \lambda_j F_j(x)\right)^2 \leq 2\sum_{x\in D(n)} F_j^2(x') + 2\lambda_j^2 \sum_{x\in D(n)} F_j^2(x).$$

But

$$\sum_{x\in D(n)} F_j^2(x') = (k-1)\sum_{x\in D(n-1)} F_j^2(x)$$

if $n \geq 2$, so that

$$\sum_{x\in D(n)} \left(F_j(x') - \lambda_j F_j(x)\right)^2 \leq 2(k-1)\sum_{x\in D(n-1)} F_j^2(x) + 2\lambda_j^2 \sum_{x\in D(n)} F_j^2(x).$$

Hence, equation (8.2) gives

$$\|\mathbf{Z}_j^{(n)}\|^2 \le \frac{(k + 2\lambda_j^2 - 1)\sum_{x \in D(n)} F_j^2(x) + 2(k-1)\sum_{x \in D(n-1)} F_j^2(x)}{\sum_{r=0}^{n} \phi_r},$$

which, together with the fact that

$$\begin{aligned}\sum_{x \in D(r)} F_j^2(x) &= \sum_{i=1}^{h} \sum_{x \in S_r^{(i)}} F_j^2(i) \\ &= \sum_{i=1}^{h} |S_r^{(i)}| F_j^2(i) \\ &= \phi_r(j),\end{aligned}$$

implies that

$$\begin{aligned}\|\mathbf{Z}_j^{(n)}\|^2 &\le \frac{(k + 2\lambda_j^2 - 1)\phi_n(j) + 2(k-1)\phi_{n-1}(j)}{\sum_{r=0}^{n} \phi_r(j)} \\ &\le (k + 2k^2 - 1)\left(\frac{\phi_{n-1}(j) + \phi_n(j)}{\sum_{r=0}^{n} \phi_r(j)}\right),\end{aligned}$$

from which the result required follows at once. ∎

To simplify this condition somewhat, let $\mathrm{supp}(\mathbf{f}_j)$ denote the (non-empty) subset of vertices of Γ on which $\mathbf{f}_j$ is non-zero, and define

$$M(\mathbf{f}_j) = \max\{\, \mathbf{f}_j^2(v_i) \mid 1 \le i \le h \,\}$$

$$m(\mathbf{f}_j) = \min\{\, \mathbf{f}_j^2(v_i) \mid v_i \in \mathrm{supp}(\mathbf{f}_j) \,\}.$$

Theorem 8.7. *For $n = 1, 2, \ldots$ define the function*

$$\psi^{(n)}(X) = \left(\frac{\sum_{i \in X}\left(|S_i^{(n)}| + |S_i^{(n-1)}|\right)}{\sum_{i \in X}\sum_{r=0}^{n} |S_i^{(r)}|}\right)$$

for any $\emptyset \neq X \subseteq V$. Then the eigensolution $(\mathbf{f}_j, \lambda_j)$ of $\Gamma = (V, E)$ lifts to T^k if

$$\liminf_{n\to\infty} \psi^{(n)}(\mathrm{supp}(\mathbf{f}_j)) = 0.$$

Proof. It is clear that

$$m(\mathbf{f}_j) \sum_{i\in \mathrm{supp}(\mathbf{f}_j)} |S_i^{(r)}| \leq \phi_r(j) \leq M(\mathbf{f}_j) \sum_{i\in \mathrm{supp}(\mathbf{f}_j)} |S_i^{(r)}|,$$

so that

$$\left(\frac{m(\mathbf{f}_j)}{M(\mathbf{f}_j)}\right) \psi^{(n)}(\mathrm{supp}(\mathbf{f}_j)) \leq \frac{\phi_n(j) + \phi_{n-1}(j)}{\sum_{r=0}^{n} \phi_r(j)} \leq \left(\frac{M(\mathbf{f}_j)}{m(\mathbf{f}_j)}\right) \psi^{(n)}(\mathrm{supp}(\mathbf{f}_j)).$$

The theorem follows from Corollary 8.6 and the fact that $m(\mathbf{f}_j)$ and $M(\mathbf{f}_j)$ are positive constants independent of n. ▪

Because of the way in which the quotient map ϕ was defined, the number $|S_i^{(n)}|$ is simply the number of reduced walks in Γ of length n with origin v and terminus v_i. Determination of these numbers for all n and all i such that $v_i \in \mathrm{supp}(\mathbf{f}_j)$ would enable us to use Theorem 8.7 to test for whether the eigensolution $(\mathbf{f}_j, \lambda_j)$ lifted to T^k or not. If it did, then Lemma 8.2 would imply immediately that $|\lambda| \leq 2\sqrt{k-1}$, so that if a lifting eigensolution could be found for all eigenvalues in $\mathrm{spec}(\Gamma) \setminus \{k, -k\}$ then we would have proved that the graph Γ was Ramanujan. However, the determination of the constants $|S_i^{(n)}|$ is, in general, not easy, so we now proceed to try and find ways in which to cut down the number of such evaluations.

Definition 8.8. *Define the functions Ψ_l, $\Psi_u : \mathcal{P}V \to [0,1]$ (where $\mathcal{P}V$ is the power set of the vertex set of Γ) by*

$$\Psi_l(X) = \begin{cases} \liminf_{n\to\infty} \psi^{(n)}(X) & (\emptyset \neq X \subseteq V) \\ 0 & (X = \emptyset) \end{cases}$$

and

$$\Psi_u(X) = \begin{cases} \limsup_{n\to\infty} \psi^{(n)}(X) & (\emptyset \neq X \subseteq V) \\ 0 & (X = \emptyset), \end{cases}$$

where $\psi^{(n)}(X)$ is as defined in the statement of Theorem 8.7. ∎

From this definition it is clear that $0 \le \Psi_l(X) \le \Psi_u(X) \le 1$ for all $X \subseteq V$. In future we will write $\Psi_l(x)$, $\Psi_u(x)$ when $X = \{x\}$.

Lemma 8.9. *Let $\mathbf{x}_1, \ldots, \mathbf{x}_r$ $(r \ge 2)$ be linearly independent elements of $\mathbf{R}^h$ (where $\mathbf{R}$ is the set of real numbers). Then there exist $\alpha_1, \ldots, \alpha_r \in \mathbf{R}$, not all zero, such that*

$$\alpha_1\mathbf{x}_1 + \cdots + \alpha_r\mathbf{x}_r = \underbrace{(0,\ldots,0)}_{r-1} \oplus \underbrace{\mathbf{y}}_{h-r+1}.$$

Proof. Induct on r. The case $r = 2$ is easily seen to be true, so suppose that it holds for some $r \ge 2$ and let $\mathbf{x}_1, \ldots, \mathbf{x}_{r+1}$ be linearly independent elements of $\mathbf{R}^h$. Then, by induction, there exist $\alpha_1, \ldots, \alpha_r \in \mathbf{R}$, not all zero, such that

$$\alpha_1\mathbf{x}_1 + \cdots + \alpha_r\mathbf{x}_r = \underbrace{(0,\ldots,0)}_{r-1} \oplus \underbrace{\mathbf{y}}_{h-r+1} \tag{8.3}$$

and we may suppose, without loss of generality, that $\alpha_1 \neq 0$. Similarly there exist $\beta_2, \ldots, \beta_{r+1} \in \mathbf{R}$, not all zero, such that

$$\beta_2\mathbf{x}_2 + \cdots + \beta_{r+1}\mathbf{x}_{r+1} = \underbrace{(0,\ldots,0)}_{r-1} \oplus \underbrace{\mathbf{z}}_{h-r+1}. \tag{8.4}$$

It is clear that $\mathbf{y}$ and $\mathbf{z}$ are independent elements of $\mathbf{R}^{h-r+1}$, for if $\mathbf{y} + \alpha\mathbf{z} = \mathbf{0}$ then

$$\alpha_1\mathbf{x}_1 + \sum_{i=2}^{r}(\alpha_i + \alpha\beta_i)\mathbf{x}_i + \alpha\beta_{r+1}\mathbf{x}_{r+1} = \mathbf{0},$$

which is not possible because $\alpha_1 \neq 0$ and the elements $\mathbf{x}_i$ $(i = 1, \ldots, r+1)$ are linearly independent.

Let $\mathbf{y} = (y_1, \ldots, y_{h-r+1})$, and define $\mathbf{z}$ similarly. Then if $y_1 = 0$ or $z_1 = 0$ the result follows immediately from (8.3) and (8.4). So we may suppose that neither of them are zero, in which case there exists a non-zero $\gamma \in \mathbf{R}$ such that $\mathbf{y} + \gamma\mathbf{z} = (0) \oplus \mathbf{w}$ with $\mathbf{w} \in \mathbf{R}^{h-r}$. Then it is clear that

$$\alpha_1\mathbf{x}_1 + \cdots + \alpha_r\mathbf{x}_r + \gamma(\beta_2\mathbf{x}_2 + \cdots + \beta_{r+1}\mathbf{x}_{r+1}) = \underbrace{(0, \ldots, 0)}_{r} \oplus \underbrace{\mathbf{w}}_{h-r},$$

which completes the induction step, and the proof. ∎

Theorem 8.10. *Let us define*

$$\eta_\Gamma = \max\{\, r \mid \exists X \subseteq V \quad \text{with} \quad |X| = r \quad \text{and} \quad \Psi_l(Y) = 0 \quad \forall Y \subseteq X\,\}.$$

Then if the eigenvalue λ of Γ has multiplicity at least $h - \eta_\Gamma + 1$, then $|\lambda| \leq 2\sqrt{k-1}$.

Proof. Suppose that X is a subset of V of size η_Γ for which $\Psi_l(Y) = 0$ for all $Y \subseteq X$. Let the eigenvalue λ have multiplicity $m \geq h - \eta_\Gamma + 1$, and suppose that $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbf{R}^h$ are m linearly independent eigenvectors having eigenvalue λ. By Lemma 8.9 it is easily seen that there exist $\alpha_1, \ldots, \alpha_m \in \mathbf{R}$, not all zero, such that if $\mathbf{x} = \sum_{i=1}^m \alpha_i\mathbf{x}_i$ then

$$\text{supp}(\mathbf{x}) \subseteq X,$$

(for, given any $m - 1$ coordinate positions, we can choose the α_i so that this linear combination is zero in those positions). If we then normalize $\mathbf{x}$ to $\mathbf{X}$, it follows that $\mathbf{X}$ is a unit eigenvector whose support Y is a subset of X, so that $\Psi_l(Y) = 0$ by choice of X with the result that the eigensolution $(\mathbf{X}, \lambda)$ lifts to T^k (using Theorem 8.7). Lemma 8.2 then implies that $|\lambda| \leq 2\sqrt{k-1}$, as required. ∎

Theorem 8.10 may be more useful than Theorem 8.7 since it only requires the multiplicity of λ and the constant η_Γ to prove that $|\lambda| \leq 2\sqrt{k-1}$. However, it is still necessary to evaluate $\psi^{(n)}(X)$ (and hence $|S_i^{(n)}|$ for all $i \in X$) over possibly quite a large number of subsets of V if we are to find η_Γ, so we will go on to find a more easily computed parameter than η_Γ.

Theorem 8.11. *Let us define*

$$\varsigma_\Gamma = \#\{\, x \in V \mid \Psi_u(x) = 0 \,\}.$$

(Note that $\Psi_u(x) = 0$ if and only if $\psi^{(n)}(x) \to 0$ as $n \to \infty$.) Then

$$\eta_\Gamma \geq \varsigma_\Gamma.$$

Proof. We set $X = \{\, x \in V \mid \Psi_u(x) = 0 \,\}$. Suppose that Y is a non-empty subset of X and that $Y = \{\, v_1, \ldots, v_t \,\}$. Then, by definition,

$$\begin{aligned}
\Psi_l(Y) &= \liminf_{n\to\infty} \psi^{(n)}(Y) \\
&= \liminf_{n\to\infty} \frac{\sum_{i=1}^t (|S_i^{(n)}| + |S_i^{(n-1)}|)}{\sum_{i=1}^t \sum_{r=0}^n |S_i^{(r)}|} \\
&\leq \liminf_{n\to\infty} \sum_{i=1}^t \frac{|S_i^{(n)}| + |S_i^{(n-1)}|}{\sum_{r=0}^n |S_i^{(r)}|} \\
&= \liminf_{n\to\infty} \sum_{i=1}^t \psi^{(n)}(v_i).
\end{aligned}$$

Using the fact that, for bounded sequences $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$ it is true that

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n,$$

then we may deduce that

$$0 \leq \liminf_{n\to\infty} \psi^{(n)}(Y) \leq \liminf_{n\to\infty} \sum_{i=1}^{t} \psi^{(n)}(v_i)$$

$$\leq \limsup_{n\to\infty} \sum_{i=1}^{t} \psi^{(n)}(v_i)$$

$$\leq \sum_{i=1}^{t} \limsup_{n\to\infty} \psi^{(n)}(v_i).$$

However $v_i \in X$ for $1 \leq i \leq t$, so that $\Psi_u(v_i) = 0$ for $1 \leq i \leq t$, or, in other words, $\limsup_{n\to\infty} \psi^{(n)}(v_i) = 0$ for $1 \leq i \leq t$. Hence

$$\liminf_{n\to\infty} \psi^{(n)}(Y) = 0,$$

so that $\Psi_l(Y) = 0$ if $Y \subseteq X$. It thus follows that $\eta_\Gamma \geq |X|$, and this completes the proof. ∎

Corollary 8.12. *If the eigenvalue λ of Γ has multiplicity at least $h - \varsigma_\Gamma + 1$, then $|\lambda| \leq 2\sqrt{k-1}$.*

Proof. Immediate from Theorems 8.10 and 8.11. ∎

Corollary 8.12 is certainly easier to apply than Theorem 8.10, for to evaluate the constant ς_Γ we need to find the limiting values of $\psi^{(n)}(v_i)$ for each singleton subset only of V. There is a particular application to Cayley graphs.

Theorem 8.13. *Let $\Gamma = \Gamma(\Omega)$ be a k-regular Cayley graph and suppose ϕ is the covering map defined as before. If χ_i is an irreducible character of Γ of degree $h_i \geq h - \varsigma_\Gamma + 1$ then the h_i^2 eigenvalues of $\Gamma(\Omega)$ associated with χ_i (in the sense of Theorem 6.3) all have modulus not exceeding $2\sqrt{k-1}$.*

Proof. A direct consequence of Corollary 8.12 and Theorem 6.3. ∎

Thus, if we know ς_Γ for such a graph, then for all irreducible representations of Γ of large enough degree, the corresponding eigenvalues of $\Gamma(\Omega)$ satisfy $|\lambda| \leq 2\sqrt{k-1}$. The problem has become that of estimating or evaluating the constant ς_Γ, and whether this is possible, or what is the efficacy of the bound obtained, remains an open question. Clearly, if all non-trivial irreducible representations of Γ have degree at least $h - \varsigma_\Gamma + 1$, then $\Gamma(\Omega)$ would be Ramanujan, for any finite generating set Ω. Failing that, it may be possible to evaluate or estimate the eigenvalues of $\Gamma(\Omega)$ arising from representations of 'small' degree, and so use Theorem 8.13 as a means of estimating the gap of the graph $\Gamma(\Omega)$.

Chapter Nine

Expanders and Free Subgroups of SL(2,Z)

9.1 Introduction

In this final Chapter we turn our attention to expander graphs (see Chapter One for a definition) and, using the techniques of Buck [**Bu**], deploy free subgroups of $SL(2, \mathbf{Z})$ to construct infinitely many families of linear expanders. In [**LPS**] an explicit construction is given, for each prime p that is congruent to 1 modulo 4, of a family of $(p+1)$-regular non-bipartite Ramanujan graphs whose orders tend to infinity. These may be used to produce an infinite number of families of linear expanders as we now show.

Definition 9.1. *An (n, k, λ)-diffuser is a doubly-stochastic $n \times n$ matrix M, with at most k non-zero entries in each row and column, such that*

$$\lambda = \max \frac{||M\mathbf{x}||}{||\mathbf{x}||}$$

taken over all non-zero vectors $\mathbf{x} = (x_1, \ldots, x_n)$ with $\sum_{i=1}^{n} x_i = 0$, and where $||\mathbf{x}|| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$. The value λ is known as the diffusion coefficient of M. ∎

In particular, we call a k-regular graph G of order n an (n, k, λ)-diffuser if its adjacency matrix A is such that $\frac{1}{k}A$ is an (n, k, λ)-diffuser in the above sense. Note that if a k-regular graph is an (n, k, λ)-diffuser then it is an $(n, k, k(1-\lambda))$-enlarger. The proof of the following theorem, which shows how diffusers may be used to make expanders, is due to Buck [**Bu**].

Lemma 9.2. *Let G be an (n, k, λ)-diffuser. Then the bipartite double cover of G is an $(n, k, 1-\lambda^2)$-expander.*

Proof. Let $G = (V, E)$ be an (n, k, λ)-diffuser, and suppose that for any $X \subseteq V$ we define the function

$$f_X(v) = \begin{cases} \frac{1}{|X|} & (v \in X); \\ 0 & (v \notin X). \end{cases}$$

Then $\|f_X\| = |X|^{-\frac{1}{2}}$ and $\sum_{v \in V} f_X(v) = 1$. If A is the adjacency matrix of G then it is clear that

$$Af_X(v) = \sum_{\{w,v\} \in E} f_X(w),$$

so that $\frac{1}{k} \sum_{v \in V} Af_X(v) = 1$. Let $\Gamma(X)$ denote (as usual) the set of vertices of G adjacent to some vertex in X, and consider the set of all real functions on V with support contained in $\Gamma(X)$ and which satisfy $\sum_{v \in V} f(v) = 1$. It is easy to see that both $f_{\Gamma(X)}$ and $\frac{1}{k} Af_X$ are in this set, and (using the Cauchy-Schwarz inequality) that $f_{\Gamma(X)}$ has the least norm of all such functions. Hence

$$\begin{aligned} \frac{1}{|\Gamma(X)|} = \|f_{\Gamma(X)}\|^2 &\leq \left\| \frac{1}{k} Af_X \right\|^2 \\ &= \frac{1}{k^2} \left\| A \left(\frac{1}{n} + \left(f_X - \frac{1}{n} \right) \right) \right\|^2 \\ &= \frac{1}{n} + \frac{1}{k^2} \left\| A \left(f_X - \frac{1}{n} \right) \right\|^2. \end{aligned}$$

Now $\sum_{v \in V} (f_X - \frac{1}{n})(v) = 0$ so that

$$\begin{aligned} \frac{1}{k^2} \left\| A \left(f_X - \frac{1}{n} \right) \right\|^2 &\leq \lambda^2 \left\| f_X - \frac{1}{n} \right\|^2 \\ &= \lambda^2 \left(\frac{1}{|X|} - \frac{1}{n} \right). \end{aligned}$$

Hence we have

$$\begin{aligned} \frac{1}{|\Gamma(X)|} &\leq \frac{1}{n} + \lambda^2 \left(\frac{1}{|X|} - \frac{1}{n} \right) \\ &= \left(1 - (1 - \lambda^2) \left(1 - \frac{|X|}{n} \right) \right) \frac{1}{|X|}. \end{aligned}$$

Consequently

$$|\Gamma(X)| \geq \left(1 - \left(1 - \lambda^2\right)\left(1 - \frac{|X|}{n}\right)\right)^{-1} |X|$$

$$\geq \left(1 + \left(1 - \lambda^2\right)\left(1 - \frac{|X|}{n}\right)\right) |X|,$$

so that the bipartite double cover of G is an $(n, k, 1 - \lambda^2)$-expander, as required. ∎

It follows from the above result that the graphs constructed in [**LPS**], being *non-bipartite* Ramanujan graphs, are $(n, p+1, 2\sqrt{p}/(p+1))$-diffusers so that their bipartite double covers are $(n, p+1, 1 - \frac{4p}{(p+1)^2})$-expanders. Thus, for each prime p congruent to 1 modulo 4 there exists a family of linear expanders of valency $k = p+1$ and expansion $1 - O(k^{-1})$. Thus we may find a family with expansion that is arbitrarily close to 1, provided that we choose the valency to be large enough.

In this chapter we shall exhibit, for $r = 2, 3, \ldots$, a family F_r of linear expanders such that the valency of F_r is $O(r^3)$ and the expansion is $1 - O(r^{-1})$. In other words, for this sequence of families the expansion is $1 - O(k^{-\frac{1}{3}})$ where k is the valency. Hence, in comparison with the Ramanujan graphs, the convergence of the expansion to 1 is much slower. However, to the author's knowledge, this is the only other explicit example of a sequence of families of linear expanders with expansion converging to 1, and the proof is more elementary than that in [**LPS**], since it does not require the proof of a special case of the Ramanujan Conjecture.

9.2 Diffusion Operators

If G is an infinite group generated by the k-set Ω (which is closed under inversion), and $L^2(G)$ is the space of complex-valued square-summable functions on G, then we define the diffusion operator M_G on this space by

$$(M_G f)(g) = \frac{1}{k} \sum_{\omega \in \Omega} f(\omega^{-1} g)$$

for any $f \in L^2(G)$. We then define the operator norm $\|M_G\|$ by

$$\|M_G\| = \sup_{\substack{f \in L^2(G) \\ f \neq 0}} \frac{\|M_G f\|}{\|f\|}$$

where $\|f\| = \left(\sum_{g \in G} |f(g)|^2\right)^{\frac{1}{2}}$. (In future all operator norms will be assumed to be derived in this way from the 2-norm on the underlying function space.) In fact (since G is countable) we may restrict $f \in L^2(G)$ to just those functions which have finite support, for any $f \in L^2(G)$ can be approximated by a sequence $\{f_n\}_{n=1}^{\infty}$ of such functions so that $\|f - f_n\| \to 0$ as $n \to \infty$. Then

$$\begin{aligned} \left| \|M_G f_n\| - \|M_G f\| \right| &\leq \|M_G(f - f_n)\| \\ &\leq \|M_G\| \times \|f - f_n\| \end{aligned}$$

so that $\|M_G f_n\| \to \|M_G f\|$ as $n \to \infty$.

From now on we set $G = SL(2, \mathbf{Z})$, the group of all 2×2 integer matrices with determinant 1. It is well known (for example [**MKS**, Section 2.3]) that the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

freely generate a subgroup of $SL(2, \mathbf{Z})$. This observation yields the following result.

Lemma 9.3. *For any finite subset $S \subseteq \{1, 2, \ldots\}$ the set of matrices*

$$\Psi(S) = \{\, A^s B^s \mid s \in S \,\}$$

freely generates a subgroup $\Gamma(S)$ of $SL(2, \mathbf{Z})$.

Proof. If $F = \langle x, y \rangle$ is a free group on the generators x and y, then there is no non-trivial relation between any of the elements $xy, x^2y^2, x^3y^3, \ldots$ [**MKS**, Section 1.4

Exercise 12] so that, for any finite subset $S \subseteq \{1, 2, \ldots\}$ the elements $\{ x^s y^s \mid s \in S \}$ freely generate a subgroup of F. The result follows upon setting $x = A$, $y = B$. ∎

It follows from this result that the Cayley graph of the group $\Gamma(S)$ with respect to the generating set $\Omega(S) = \Psi(S) \cup \Psi(S)^{-1}$ is the infinite $2|S|$-regular tree so that, by a result of Kesten [**Ke**], we have

$$\|M_{\Gamma(S)}\| = \frac{2\sqrt{2|S|-1}}{2|S|}$$

$$= \frac{\sqrt{2|S|-1}}{|S|}.$$

The group $SL(2, \mathbf{Z})$ acts on the lattice (minus the origin) $\mathcal{L} = \mathbf{Z} \times \mathbf{Z} \setminus \{\mathbf{0}\}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (x, y) \mapsto (ax + by, cx + dy).$$

There is a corresponding diffusion operator $M = M_{\Gamma(S),\mathcal{L}}$ on the space $L^2(\mathcal{L})$ given by

$$(Mf)(x) = \frac{1}{2|S|} \sum_{\omega \in \Omega(S)} f(\omega^{-1} x)$$

for any $f \in L^2(\mathcal{L})$. We need the following result to show that the norm of this operator (defined in the usual way) equals that of $M_{\Gamma(S)}$.

Lemma 9.4. *Let G be an infinite group generated by the finite set Ω $(= \Omega^{-1})$, and suppose that it acts transitively on the set A with the point stabiliser of some element $a \in A$ being H (so that A is isomorphic to the coset space $\cos(G/H)$). Define diffusion operators on $L^2(G)$ and $L^2(A)$ respectively by*

$$M_G f(g) = \frac{1}{|\Omega|} \sum_{\omega \in \Omega} f(\omega^{-1} g)$$

$$M_A f(a) = \frac{1}{|\Omega|} \sum_{\omega \in \Omega} f(\omega^{-1}(a)).$$

Then if H is Abelian it follows that $\|M_G\| = \|M_A\|$ (both operator norms being defined in the usual way).

Proof. Buck [**Bu**, Proposition 4.1]. ■

Lemma 9.5.

$$\|M\| = \frac{\sqrt{2|S|-1}}{|S|}.$$

Proof. The stabilizer of the element $(r, 0) \in \mathcal{L}$ under the action of $SL(2, \mathbf{Z})$ described above is the Abelian group

$$K = \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \;\middle|\; \alpha \in \mathbf{Z} \right\}.$$

Now it is easy to see that, under the action of $SL(2, \mathbf{Z})$, the set $\mathcal{L}$ has orbits $A_1, A_2, \ldots$, where

$$A_r = \{ (x, y) \in \mathcal{L} \mid \quad \mathrm{hcf}(x, y) = r \}$$

for $r = 1, 2, \ldots$. Hence, for any $(x, y) \in \mathcal{L}$ there exists a positive integer r and an element $\gamma \in SL(2, \mathbf{Z})$ such that $(x, y) = \gamma(r, 0)$. Consequently, the stabilizer of (x, y) under the action of $\Gamma(S)$ is the Abelian group $\Gamma(S) \cap \gamma K \gamma^{-1}$. Let $M_{(x,y)}$ denote the operator which is the restriction of M to those functions in $L^2(\mathcal{L})$ whose support is contained in the orbit of (x, y) (under the action of $\Gamma(S)$ on $\mathcal{L}$). Then, by applying Lemma 9.4 with $G = \Gamma(S)$, $\Omega = \Omega(S)$, A being the orbit of (x, y) and $H = \Gamma(S) \cap \gamma K \gamma^{-1}$, we deduce that

$$\|M_{(x,y)}\| = \|M_{\Gamma(S)}\|$$

$$= \frac{\sqrt{2|S|-1}}{|S|} \qquad \forall (x, y) \in \mathcal{L}.$$

It only remains to prove that $\|M_{(x,y)}\| = \|M\|$. As we have previously pointed out, we need consider only the functions $f \in L^2(\mathcal{L})$ with finite support. Given such an f, there are only a finite number of orbits on which f is non-zero. Let (x_i, y_i) and f_i $(1 \le i \le t)$ be representatives of these orbits and the restriction of f to these orbits, respectively. Then

$$\frac{\|Mf\|^2}{\|f\|^2} = \frac{\|M_{(x_1,y_1)}f_1\|^2 + \cdots + \|M_{(x_t,y_t)}f_t\|^2}{\|f_1\|^2 + \cdots + \|f_t\|^2}$$

$$\leq \max_{1\leq i\leq t} \frac{\|M_{(x_i,y_i)}f_i\|^2}{\|f_i\|^2}$$

$$\leq \left(\frac{\sqrt{2|S|-1}}{|S|}\right)^2,$$

so that $\|M\| \leq (\sqrt{2|S|-1}/|S|)$. On the other hand, if f is such that

$$\frac{\|M_{(x_1,y_1)}f\|}{\|f\|} = \|M_{(x_1,y_1)}\|,$$

then by extending the domain of f to all of $\mathcal{L}$ (simply by defining it to be zero outside the orbit of (x_1, y_1)) we see that

$$\frac{\|Mf\|}{\|f\|} = \frac{\|M_{(x_1,y_1)}f\|}{\|f\|} = \frac{\sqrt{2|S|-1}}{|S|}.$$

Hence $\|M\| \geq (\sqrt{2|S|-1}/|S|)$, so that

$$\|M\| = \frac{\sqrt{2|S|-1}}{|S|}$$

as required. ∎

There is another action of $SL(2, \mathbf{Z})$ which will be of use to us, namely that on the 2-torus $T^2 = \mathbf{R}^2/\mathbf{Z}^2$, defined in the same way as the action upon $\mathcal{L}$. The diffusion operator $M_{T^2,S}$ on $L^2(T^2)$ corresponding to the action of $\Gamma(S)$ on T^2 is given by

$$(M_{T^2,S}f)(x) = \frac{1}{2|S|} \sum_{\omega\in\Omega(S)} f(\omega^{-1}x).$$

We denote by $M^0_{T^2,S}$ the restriction of this operator to the subspace

$$L^2_0(T^2) = \left\{ f \in L^2(T^2) \,\middle|\, \int_{T^2} f(y)\, d\mu(y) = 0 \right\}$$

where μ is the (normalized) Lebesgue measure on T^2. We shall use the quantity $\|M^0_{T^2,S}\|$ (which is defined in the usual way from the norm $\|f\|^2 = \int_{T^2} |f(y)|^2\, d\mu(y)$) to estimate the expansion of our expanding graphs.

9.3 Expanding Graphs from $\Gamma(S)$

Let τ be the automorphism of $SL(2, \mathbf{Z})$ that takes a matrix to the inverse of its transpose. Then, if A and B are as defined in Section 9.2, we have $\tau(A^s B^s) = B^{-s} A^{-s}$ so that the generating set $\Omega(S)$ is closed under the action of τ. Recall that, for a function $f \in L^2(T^2)$, the Fourier transform of f is

$$\hat{f}(x) = \int_{T^2} \exp(-2\pi i \langle x, y\rangle) f(y)\, d\mu(y)$$

where $\langle x, y\rangle$ denotes the usual scalar product on T^2.

Lemma 9.6. *The Fourier transform is a surjection from $L^2_0(T^2)$ to $L^2(\mathcal{L})$.*

Proof. Parseval's identity gives

$$\int_{T^2} |f(y)|^2\, d\mu(y) = \sum_{x \in \mathbf{Z}^2} |\hat{f}(x)|^2,$$

so that $f \in L^2(T^2)$ implies that $\hat{f} \in L^2(\mathbf{Z}^2)$. Furthermore

$$\hat{f}(0) = \int_{T^2} f(y)\, d\mu(y)$$

$$= 0 \qquad \text{if} \quad f \in L^2_0(T^2)$$

so that $f \in L^2_0(T^2)$ implies that we may assume that $\hat{f} \in L^2(\mathcal{L})$. That the map is surjective is a simple consequence of the Riesz-Fischer Theorem. ∎

For $\gamma \in SL(2, \mathbf{Z})$ the operator U_γ is defined by

$$(U_\gamma f)(x) = f(\gamma^{-1}x).$$

Then, for a finite subset $S \subseteq \{1, 2, \ldots\}$, we define

$$U_S = \frac{1}{2|S|} \sum_{\omega \in \Psi(S)} \left(U_\omega + U_{\tau(\omega)}\right).$$

Then clearly

$$U_S f = \begin{cases} M^0_{T^2,S} f & \text{if } f \in L^2_0(T^2); \\ Mf & \text{if } f \in L^2(\mathcal{L}). \end{cases}$$

The following theorem (due to Buck [**Bu**]) shows that the diagram shown below commutes (where F is the Fourier transform map) :

$$\begin{array}{ccc} L^2_0(T^2) & \xrightarrow{F} & L^2(\mathcal{L}) \\ \Big\downarrow {\scriptstyle M^0_{T^2,S}} & & \Big\downarrow {\scriptstyle M} \\ L^2_0(T^2) & \xrightarrow{F} & L^2(\mathcal{L}) \end{array}$$

Theorem 9.7. *If $f \in L^2_0(T^2)$ then*

$$(\widehat{M^0_{T^2,S} f})(x) = (M\hat{f})(x).$$

Proof.

$$(\widehat{M^0_{T^2,S} f})(x) = \int_{T^2} \exp(-2\pi i \langle x, y\rangle)\left(M^0_{T^2,S} f\right)(y)\, d\mu(y)$$

$$= \frac{1}{2|S|} \sum_{\omega \in \Psi(S)} \int_{T^2} \exp(-2\pi i \langle x, y\rangle)\left(U_\omega + U_{\tau(\omega)}\right) f(y)\, d\mu(y)$$

However

$$\int_{T^2} \exp(-2\pi i\langle x,y\rangle)U_\omega f(y)\,d\mu(y) = \int_{T^2} \exp(-2\pi i\langle x,\omega(z)\rangle)U_\omega f(\omega(z))\,d\mu(z)$$

$$= \int_{T^2} \exp(-2\pi i\langle \omega^T(x),z\rangle) f(z)\,d\mu(z)$$

where ω^T denotes the transpose of ω, so that

$$\int_{T^2} \exp(-2\pi i\langle x,y\rangle)U_\omega f(y)\,d\mu(y) = \hat{f}(\omega^T(x))$$

$$= (U_{\tau(\omega)}\hat{f})(x).$$

Similarly, we have

$$\int_{T^2} \exp(-2\pi i\langle x,y\rangle)U_{\tau(\omega)} f(y)\,d\mu(y) = (U_\omega \hat{f})(x),$$

so that we deduce that

$$(\widehat{M^0_{T^2,S}f}) = \frac{1}{2|S|}\sum_{\omega\in\Psi(S)} (U_\omega + U_{\tau(\omega)})\hat{f}(x)$$

$$= (M\hat{f})(x).$$

as required. ∎

Corollary 9.8.

$$\|M^0_{T^2,S}\| = \frac{\sqrt{2|S|-1}}{|S|}.$$

Proof. If $f \in L_0^2(T^2)$ then

$$\begin{aligned}\|M^0_{T^2,S} f\|^2 &= \int_{T^2} |(M^0_{T^2,S} f)(y)|^2 \, d\mu(y) \\ &= \sum_{x \in \mathcal{L}} |(\widehat{M^0_{T^2,S} f})(x)|^2 \qquad \text{by Parseval's identity} \\ &= \sum_{x \in \mathcal{L}} |(M\hat{f})(x)|^2 \qquad \text{by Theorem 9.7} \\ &= \|M\hat{f}\|^2.\end{aligned}$$

Similarly we have $\|f\|^2 = \|\hat{f}\|^2$ so that

$$\frac{\|M^0_{T^2,S} f\|}{\|f\|} = \frac{\|M\hat{f}\|}{\|\hat{f}\|} \quad \forall f \in L_0^2(T^2)$$

and hence that $\|M^0_{T^2,S}\| = \|M\|$. The result then follows from Lemmas 9.5 and 9.6. ∎

We are now in a position to define our expanding graphs. Given the finite subset $S \subseteq \{1, 2, \ldots\}$ we define the *spreading operator* Γ for subsets $X \subseteq T^2$ by

$$\Gamma(X) = \bigcup_{\omega \in \Omega(S)} \omega(X).$$

Lemma 9.9. *For any $X \subseteq T^2$ we have*

$$\mu(\Gamma(X)) \geq \left(1 + \left(1 - \frac{1}{|S|}\right)^2 \mu(X^c)\right) \mu(X)$$

where $X^c = T^2 \setminus X$.

Proof. The proof is essentially a continuous version of that of Lemma 9.2. We set $\lambda = \|M^0_{T^2,S}\|$, $X \subseteq T^2$, and define $f_X \in L^2(T^2)$ by

$$f_X(y) = \begin{cases} \frac{1}{\mu(X)} & (y \in X); \\ 0 & (\text{otherwise}). \end{cases}$$

Then $\|f_X\| = \mu(X)^{-\frac{1}{2}}$ and $\int_{T^2} f_X(y)\,d\mu(y) = 1$. Consider the subset of $L^2(T^2)$ of functions f with support contained in $\Gamma(X)$ and which satisfy $\int_{T^2} f(y)\,d\mu(y) = 1$. Both $f_{\Gamma(X)}$ and $M_{T^2,S}f_X$ lie in this set, and it is easy to see (using the integral form of the Cauchy-Schwarz inequality) that $f_{\Gamma(X)}$ has the least norm of all such functions. Hence

$$\frac{1}{\mu(\Gamma(X))} = \|f_{\Gamma(X)}\|^2 \leq \|M_{T^2,S}f_X\|^2$$

$$= \|M_{T^2,S}\left(1 + (f_X - 1)\right)\|^2$$

$$= 1 + \|M^0_{T^2,S}(f_X - 1)\|^2$$

$$\leq 1 + \lambda^2\|f_X - 1\|^2.$$

Then, using the fact that $\|f_X - 1\|^2 = \mu(X)/\mu(X^c)$, we proceed in a manner analogous to that in the proof of Lemma 9.2 to show that

$$\mu(\Gamma(X)) \geq \left(1 + \left(1 - \lambda^2\right)\mu(X^c)\right)\mu(X).$$

The result follows at once because $\lambda = \sqrt{2|S| - 1}/|S|$. ∎

Now let n be a positive integer. Following the methods of Gabber & Galil [**GG**] and their generalisation by Buck[**Bu**], we divide the torus T^2 into n^2 subsets, of the form

$$\mathrm{Sq}(a,b) = \left\{ (x,y) \in T^2 \;\middle|\; \frac{a}{n} \leq x < \frac{a+1}{n}, \frac{b}{n} \leq y < \frac{b+1}{n} \right\},$$

for each $(a,b) \in \mathbf{Z}_n \times \mathbf{Z}_n$ (where $\mathbf{Z}_n$ is the ring of integers modulo n). For a positive integer s, we will consider the set $A^sB^s\mathrm{Sq}(a,b)$. We have

$$A^sB^s = \begin{pmatrix} 1 & 2s \\ 2s & 4s^2+1 \end{pmatrix},$$

so that

$$A^sB^s : (x, y) \mapsto (x + 2sy, 2sx + (4s^2 + 1)y).$$

Suppose that $(x, y) \in \mathrm{Sq}(a, b)$ so that, for some q in the set $\{0, 1, \ldots, 2s\}$, we have

$$\frac{a + 2sb + q}{n} \le x + 2sy < \frac{a + 2sb + q + 1}{n}.$$

Then clearly

$$\frac{2sa + (4s^2 + 1)b + 2sq}{n} \le 2sx + (4s^2 + 1)y < \frac{2sa + (4s^2 + 1)b + 2sq + 2s + 1}{n}.$$

We deduce that

$$A^sB^s\mathrm{Sq}(a, b) \subseteq \bigcup_{q=0}^{2s} \bigcup_{p=0}^{2s} \mathrm{Sq}(a + 2sb + q, 2sa + (4s^2 + 1)b + 2sq + p) \tag{9.1}$$

and similarly that

$$B^{-s}A^{-s}\mathrm{Sq}(a, b) \subseteq \bigcup_{q=0}^{2s} \bigcup_{p=0}^{2s} \mathrm{Sq}((1 + 4s^2)a - 2sb + 2sq - p, -2sa + b - q). \tag{9.2}$$

Given a finite subset $S \subseteq \{1, 2, \ldots\}$, we construct a finite bipartite graph as follows. Let the vertices of each bipartite block be labelled with the elements of $\mathbf{Z}_n \times \mathbf{Z}_n$. Then, for each $s \in S$, join each vertex (a, b) in the first block to the vertices of the second block given by

$$\sigma_{p,q}(a, b) = (a + 2sb + q, 2sa + (4s^2 + 1)b + 2sq + p)$$

and

$$\phi_{p,q}(a, b) = ((1 + 4s^2)a - 2sb + 2sq - p, -2sa + b - q)$$

for $0 \le p, q \le 2s$.

Theorem 9.10. *For any finite subset $S \subseteq \{1, 2, \ldots\}$ the family $\{G_n(S)\}_{n=1}^{\infty}$ is a family of linear expanders of valency $2\sum_{s\in S}(2s+1)^2$ and expansion $\left(1 - \frac{1}{|S|}\right)^2$.*

Proof. Clearly for each $s \in S$ there are $2(2s+1)^2$ edges issuing from each vertex in the first block, whence the expression for the valency.

To calculate the expansion of $G_n(S)$ we take a subset X of the vertices in the first block, and let $\Gamma(X)$ be its set of neighbours in $G_n(S)$. (We will also use Γ to denote the spreading operator on T^2, but it will be clear from the context which definition of Γ is being used).

Let

$$\mathrm{Sq}(X) = \bigcup_{(a,b)\in X} \mathrm{Sq}(a, b).$$

Then, by definition of adjacency in $G_n(S)$ and the equations (9.1) and (9.2), we have

$$\Gamma(\mathrm{Sq}(X)) \subseteq \mathrm{Sq}(\Gamma(X))$$

so that

$$|\Gamma(X)| = n^2 \mu(\mathrm{Sq}(\Gamma(X)))$$

$$\geq n^2 \mu(\Gamma(\mathrm{Sq}(X))).$$

Thus, by the result of Lemma 9.9 applied to the subset $\mathrm{Sq}(X) \subseteq T^2$, we have

$$|\Gamma(X)| \geq n^2 \left(1 + \left(1 - \frac{1}{|S|}\right)^2 \mu(\mathrm{Sq}(X)^c)\right) \mu(\mathrm{Sq}(X))$$

$$= \left(1 + \left(1 - \frac{1}{|S|}\right)^2 \left(1 - \frac{|X|}{n^2}\right)\right) |X|.$$

Hence $G_n(S)$ is an $(n^2, 2\sum_{s\in S}(2s+1)^2, \left(1 - \frac{1}{|S|}\right)^2)$-expander, and the result follows. ∎

Clearly, to minimise the valency of $G_n(S)$ whilst keeping the expansion fixed, we should choose subsets of the form $S_r = \{1, 2, \ldots, r\}$. We then have the following result.

Corollary 9.11. *Let $k(r) = 2r(4r^2 + 12r + 11)/3$. Then for each integer $r \geq 2$ there exists a family of linear expanders of valency $k(r)$ and expansion $\left(1 - \frac{1}{r}\right)^2$.*

Proof. The graph $G_n(S_r)$ has valency

$$2 \sum_{s \in S_r} (2s+1)^2 = 8 \sum_{s=1}^{r} s^2 + 8 \sum_{s=1}^{r} s + 2r$$

$$= \frac{4}{3} r(r+1)(2r+1) + 4r(r+1) + 2r$$

$$= k(r).$$

Hence $G_n(S_r)$ is an $(n^2, k(r), \left(1 - \frac{1}{|S|}\right)^2)$-expander, so that $\{G_n(S_r)\}_{n=1}^{\infty}$ is a family of linear expanders of valency $k(r)$ and expansion $\left(1 - \frac{1}{r}\right)^2$. ∎

It is a simple matter to estimate the density of the corresponding families of linear superconcentrators using Theorem 3 in [**GG**]. For $r = 2$, the family $\{G_n(S_2)\}_{n=1}^{\infty}$ yields superconcentrators of density equal to 1252, and this value increases with r, so that the in this sense the graphs we have described have poor expansion properties (compare the value for the density of 58 obtained from the graphs derived in [**LPS**]). However, as we have said, we can achieve expansion arbitrarily close to 1 by choosing the valency to be large enough, which has only been achieved elsewhere by using the Ramanujan graphs of Lubotzky, Phillips and Sarnak.

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